ORIGINAL ARTICLE

Operational Matrix Basic Spline Wavelets of Derivative for Linear Optimal Control Problem

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Abstract: The main importance goal in this paper is studying the interesting properties of basic spline wavelets functions (BSWFs) and derived some new basic formulations of them. The important operational matrix is devoted in two ways, the first one is the derivative of BSWFs in terms of the lower order of BSWFs while the second is the derivative of BSWFs in terms of the same order of BSWFs. The expression formula for the operational matrix is determined for different orders. In addition an useful formulas concerning the power function and BSMSFs are also presented. The polynomials and wavelets expansions together with operational matrices can be employed to solve problems in applied science and other fields of approximation theory. In this work, two optimal control problem are tested with the aid of operational matrix of derivative for BSWFs with satisfactory results.

Keywords: Basic spline wavelets; Operational matrix; Linear optimal control problem

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1. Introduction

Wavelets have been successfully utilized in scientific and engineering problems. Basic spline scaling and wavelets functions play an important role in mathematics; they have been utilized in the solution of differential equations, integral equations and approximation theory. In particular basic spline wavelets have been applied in the approximation of linear and nonlinear Volterra and Fredholm integral equations[1-2]. One of the most important algorithms for treating many problems approximately is based on using operational matrix of derivatives. The advantages of operational matrices are to convert the original problem to a system of algebraic equations and then the differentiation will be eliminating with the aide of operational matrix of derivative. In a result, the complexity reduction can be obtained. About the applications of operational matrices, there are some papers for example[3-6]. In particular, some researchers applied different basis polynomials and functions for solving optimal control problems, such as, shifted Chebyshev polynomials[7], Chebyshev wavelets[8], Legendre orthonormal basis[9], Tayler wavelets[10], interpolating scaling functions[11], third kind Chebyshev wavelets[12], Bernstein and orthonormal Bernstein polynomials[13-19].

In this paper, novel approach based on BSMSFs operational matrices with their properties is applied for approximate solution of linear optimal control problem.

2. Basic spline Wavelets Functions

The basic spline wavelets functions $\delta_{in}^{m}(t)$ can be constructed on the interval $0 \le t < 1$ as

$$\delta_{in}^{m}(t) = \begin{cases} \left(\sqrt{2}\right)^{k} OBS_{i}^{m}\left(2^{k}t - n\right) & \frac{n}{2^{k}} \le t < \frac{n+1}{2^{k}} \\ 0 & otherwise \end{cases} \tag{1}$$

where $i = 0, 1, \ldots, 2^{k-1}$ and the four arguments n, k, m, t are

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(1) The translation argument $n = 0, 1, \ldots, m$

(2) The number of partitions on $[0, 1]$, k is any positive integer

(3) The normalized time t

(4) The order of orthonormal B-spline function on $[0, 1]$ is m.

For $k = 1$, $m = 1, n = 0,1$ and $i = 0,1$

$$\delta_{00}^1 = \sqrt{6}(1-2t)$$
$$\delta_{01}^1 = \sqrt{2}(6t-1)\ , 0 \leq \mathrm{t} \leq \tfrac{1}{2}$$

and

$$\delta_{10}^1 = \sqrt{6}(2-2t)$$
$$\delta_{11}^1 = \sqrt{2}(6t-4)\ , \tfrac{1}{2} \leq \mathrm{t} \leq 1$$

For $k = 1$, $m = 2, n = 0,1,2$ and $i = 0,1$

$$\delta_{00}^2 = \sqrt{10}(1-2t)^2$$
$$\delta_{01}^2 = \sqrt{6}(-20t^2+12t-1)\ , 0 \leq \mathrm{t} \leq \tfrac{1}{2}$$
$$\delta_{02}^2 = \sqrt{2}(40t^2-16t+1)$$

and

$$\delta_{10}^2 = \sqrt{10}(2-2t)^2$$
$$\delta_{11}^2 = \sqrt{6}(-20\mathrm{t}^2+32\mathrm{t}-12)\ , \tfrac{1}{2} \leq t \leq 1$$
$$\delta_{12}^2 = \sqrt{2}(40\mathrm{t}^2-56\mathrm{t}+19)$$

For $k = 1$, $m = 3, n = 0, 1, 2, 3$ and $i = 0, 1$

$$\delta_{00}^3 = \sqrt{14}(1-2t)^3$$
$$\delta_{01}^3 = \sqrt{10}(1-2t)^2(14t-1)\ , 0 \leq t \leq \tfrac{1}{2}\ t \leq \tfrac{1}{2}$$
$$\delta_{02}^3 = \sqrt{6}(1-2t)(84t^2-24t+1)$$
$$\delta_{03}^3 = \sqrt{2}(280t^3-180t^2+30t-1)$$

and

$$\delta_{10}^3 = \sqrt{14}(2-2t)^3$$
$$\delta_{11}^3 = \sqrt{10}(2-2t)^2(14t-8) \quad , \quad \tfrac{1}{2} \leq t \leq 1$$
$$\delta_{12}^3 = \sqrt{6}(2-2t)(84t^2-108t+34)$$
$$\delta_{13}^3 = \sqrt{2}(280t^3-600t^2+420t+96)$$

For $k = 1$, $m = 2, n = 0, 1, 2$ and $i = 0, 1$

$$\delta_{00}^4 = 3\sqrt{2}(1-2t)^4$$
$$\delta_{01}^4 = \sqrt{14}(1-2t)^3(18t-1)$$
$$\delta_{02}^4 = \sqrt{10}(1-2t)^2(144t^2-32t+1) \quad , \quad 0 \leq t \leq \tfrac{1}{2}$$
$$\delta_{03}^4 = \sqrt{6}(1-2t)(672t^3-336t^2+42t-1)$$
$$\delta_{04}^4 = \sqrt{2}(2016t^4-1792t^3+252t^2-48t+1)$$

and

$$\delta_{10}^4 = 3\sqrt{2}(2-2t)^4$$
$$\delta_{11}^4 = \sqrt{14}(2-2t)^3(18t-10)$$
$$\delta_{12}^4 = \sqrt{10}(2-2t)^2(144t^2-176t+53)\ , \quad \tfrac{1}{2} \leq t \leq 1$$
$$\delta_{13}^4 = \sqrt{6}(1-2t)(672t^3-1344t^2+882t-190)$$
$$\delta_{04}^4 = \sqrt{2}(2016t^4-5824t^3+596t^2-2652t+438)$$

3. Operational Matrix of Derivative for BSWFs

This section gives the constructing operational matrix of derivative for BSWFs.

For $m = 2$

$$\dot{\delta}_{00}^2 = -\frac{4\sqrt{5}}{\sqrt{3}}\delta_{00}^1$$
$$\dot{\delta}_{01}^2 = 8\delta_{00}^1 - 4\sqrt{3}\delta_{01}^1 \quad , \quad 0 \leq t \leq \tfrac{1}{2}$$
$$\dot{\delta}_{02}^2 = -\frac{4}{\sqrt{3}}\delta_{00}^1 + 12\delta_{01}^1$$

and

$$\dot{\delta}_{10}^2 = -\frac{4\sqrt{5}}{\sqrt{3}}\delta_{10}^1$$
$$\dot{\delta}_{11}^2 = 8\delta_{10}^1 - 4\sqrt{3}\delta_{11}^1 \quad , \quad \tfrac{1}{2} \leq t \leq 1$$
$$\dot{\delta}_{12}^2 = -\tfrac{4}{\sqrt{3}}\delta_{10}^1 + 12\delta_{11}^1$$

One can write the above equations as

$$\dot{\delta}^2(\mathrm{t}) = D\delta^1(\mathrm{t})$$

where

$$\dot{\delta}^2(t) = [\dot{\delta}_{00}^2 \quad \dot{\delta}_{01}^2 \quad \dot{\delta}_{02}^2 \quad \dot{\delta}_{10}^2 \quad \dot{\delta}_{11}^2 \quad \dot{\delta}_{12}^2]^T$$
$$\delta^1(t) = [\delta_{00}^1 \quad \delta_{01}^1 \quad \delta_{10}^1 \quad \delta_{11}^1]^T$$

and the operational matrix D is a 6×4 matrix $D = \begin{pmatrix} D_1 & 0 \\ 0 & D_1 \end{pmatrix}$

where $D_{1=} \begin{pmatrix} -\frac{4\sqrt{5}}{\sqrt{3}} & 0 \\ 8 & -4\sqrt{2} \\ -\frac{4}{\sqrt{3}} & 12 \end{pmatrix}$

For $m = 2$

$$\dot{\delta}_{00}^3 = -\frac{6\sqrt{7}}{\sqrt{5}}\delta_{00}^2$$
$$\dot{\delta}_{01}^3 = 12\delta_{00}^2 - \tfrac{6\sqrt{5}}{\sqrt{3}}\delta_{01}^2\ , \quad 0 \leq t \leq \tfrac{1}{2}$$
$$\dot{\delta}_{02}^3 = -\frac{28\sqrt{3}}{3\sqrt{5}}\delta_{00}^2 + 10\delta_{01}^2 - \frac{20\sqrt{3}}{\sqrt{3}}\delta_{02}^2$$
$$\dot{\delta}_{03}^3 = \frac{10}{\sqrt{5}}\delta_{00}^2 + 20\delta_{02}^2$$

and

$$\dot{\delta}_{10}^3 = -\tfrac{6\sqrt{7}}{\sqrt{5}}\delta_{10}^2 \dot{\delta}_{11}^3 = 12\delta_{10}^2 - \tfrac{6\sqrt{5}}{\sqrt{3}}\delta_{11}^2, \quad \tfrac{1}{2} \leq t \leq 1$$
$$\dot{\delta}_{12}^3 = -\frac{28\sqrt{3}}{3\sqrt{5}}\delta_{10}^2 + 10\delta_{11}^2 - \frac{20\sqrt{3}}{\sqrt{3}}\delta_{12}^2$$
$$\dot{\delta}_{13}^3 = \frac{10}{\sqrt{5}}\delta_{10}^2 + 20\delta_{12}^2$$

One can write the above equations as

$$\dot{\delta}^3(t) = D\delta^2(t)$$

where

$$\dot{\delta}^3(t) = [\dot{\delta}_{00}^3 \quad \dot{\delta}_{01}^3 \quad \dot{\delta}_{02}^3 \quad \dot{\delta}_{03}^3 \quad \dot{\delta}_{10}^3 \quad \dot{\delta}_{11}^3 \quad \dot{\delta}_{12}^3 \quad \dot{\delta}_{13}^3\ ]^T$$
$$\delta^2(t) = [\delta_{00}^2 \quad \delta_{01}^2 \quad \delta_{02}^2 \quad \delta_{10}^2 \quad \delta_{11}^2 \quad \delta_{12}^2]^T$$

In this case $D_1=\begin{pmatrix} -\frac{6\sqrt{7}}{\sqrt{5}} & 0 & 0 \\ 12 & -\frac{6\sqrt{5}}{\sqrt{3}} & 0 \\ -\frac{28\sqrt{3}}{3\sqrt{5}} & 10 & -\frac{20\sqrt{3}}{3} \\ \frac{10}{\sqrt{5}} & 0 & 20 \end{pmatrix}$

$m = 4$

and

$$\dot{\delta}^4_{10} =- \frac{24}{\sqrt{7}}\delta^3_{10}$$
$$\dot{\delta}^4_{11} = 16\delta^3_{10} - \frac{8\sqrt{7}}{\sqrt{5}}\delta^3_{11}$$
$$\dot{\delta}^4_{12} =- \frac{68\sqrt{5}}{5\sqrt{7}}\delta^3_{10} + 14\delta^3_{11} - \frac{42\sqrt{5}}{5\sqrt{3}}\delta^3_{12}, \quad \frac{1}{2} \le t \le 1$$
$$\dot{\delta}^4_{13} = \frac{14\sqrt{3}}{\sqrt{7}}\delta^3_{10} - \frac{8\sqrt{3}}{\sqrt{5}}\delta^3_{11} + 12\delta^3_{12} - 10\sqrt{3}\delta^3_{13}$$
$$\dot{\delta}^4_{14} =- \frac{511}{5\sqrt{7}}\delta^3_{10} - \frac{179}{\sqrt{5}}\delta^3_{11} - \frac{789}{5\sqrt{3}}\delta^3_{12} - 33\delta^3_{13}$$

One can write the above equations as

$$\dot{\delta}^4(t) = D\delta^3(t)$$

where

$\dot{\delta}^4(t) = [\dot{\delta}^4_{00} \quad \dot{\delta}^4_{01} \quad \dot{\delta}^4_{02} \quad \dot{\delta}^4_{03} \quad \dot{\delta}^4_{04} \quad \dot{\delta}^4_{10} \quad \dot{\delta}^4_{11} \quad \dot{\delta}^4_{12} \quad \dot{\delta}^4_{13} \quad \dot{\delta}^4_{14}]^T$

$\delta^3(t) = [\delta^3_{00} \quad \delta^3_{01} \quad \delta^3_{02} \quad \delta^3_{03} \quad \delta^3_{10} \quad \delta^3_{11} \quad \delta^3_{12} \quad \delta^3_{13}]^T$

In this case, the matrix D_1 is equal to

$$\begin{pmatrix} -\frac{24}{\sqrt{7}} & 0 & 0 & 0 \\ 16 & -\frac{8\sqrt{7}}{\sqrt{5}} & 0 & 0 \\ -\frac{68\sqrt{5}}{5\sqrt{7}} & 14 & -\frac{42\sqrt{5}}{5\sqrt{3}} & 0 \\ \frac{14\sqrt{3}}{\sqrt{7}} & -\frac{8\sqrt{3}}{\sqrt{5}} & 12 & -10\sqrt{3} \\ -\frac{511}{5\sqrt{7}} & -\frac{179}{\sqrt{5}} & -\frac{789}{5\sqrt{3}} & -33 \end{pmatrix}$$

4. Operational Matrix of Derivative for BSWFs in terms of the Same Order of BSWFs

$m = 1$

$$\dot{\delta}^1_{00} =- 3\delta^1_{00} - \sqrt{3}\,\delta^1_{01}$$
$$\dot{\delta}^1_{01} = 3\sqrt{3}\delta^1_{00} + \sqrt{3}\,\delta^1_{01} \quad , \quad 0 \le t \le \frac{1}{2}$$

and

$$\dot{\delta}^1_{10} =- 3\delta^1_{10} - \sqrt{3}\,\delta^1_{11}$$
$$\dot{\delta}^1_{11} = 3\sqrt{3}\delta^1_{10} + \sqrt{3}\,\delta^1_{11} \quad , \quad \frac{1}{2} \le t \le 1$$

One can write the above equations as

$\dot{\delta}^1(t) = D\delta^1(t)$ where $D_1 = \begin{pmatrix} -3 & -\sqrt{3} \\ 3\sqrt{3} & 3 \end{pmatrix}$

$m = 2$

$$\dot{\delta}^2_{00} =- 5\delta^2_{00} - \frac{\sqrt{5}}{\sqrt{3}}\delta^2_{01}$$
$$\dot{\delta}^2_{01} = \frac{35\sqrt{3}}{3\sqrt{5}}\delta^2_{00} - 3\delta^2_{01} - \frac{8\sqrt{3}}{3}\delta^2_{02} \quad , \quad 0 \le t \le \frac{1}{2}$$
$$\dot{\delta}^2_{02} =- \frac{10}{\sqrt{5}}\delta^2_{00} + \frac{14}{\sqrt{6}}\delta^2_{01} + 8\delta^2_{02}$$

and

$$\dot{\delta}^2_{10} =- 5\delta^2_{10} - \frac{\sqrt{5}}{\sqrt{3}}\delta^2_{11}$$
$$\dot{\delta}^2_{11} = \frac{35\sqrt{3}}{3\sqrt{5}}\delta^2_{10} - 3\delta^2_{11} - \frac{8\sqrt{3}}{3}\delta^2_{12}, \quad \frac{1}{2} \le t \le 1$$
$$\dot{\delta}^2_{12} =- \frac{10}{\sqrt{5}}\delta^2_{10} + \frac{14}{\sqrt{6}}\delta^2_{11} + 8\delta^2_{12}$$

One can write the above equations as

$\dot{\delta}^2(t) = D\delta^2(t)$, where

$$D1 = \begin{pmatrix} -5 & -\frac{\sqrt{5}}{\sqrt{3}} & 0 \\ \frac{35\sqrt{3}}{3\sqrt{5}} & -3 & -\frac{8\sqrt{3}}{3} \\ -\frac{10}{\sqrt{5}} & \frac{14}{\sqrt{3}} & 8 \end{pmatrix}$$

$m = 3$

$$\dot{\delta}^3_{00} =- 7\delta^3_{00} - \frac{\sqrt{7}}{\sqrt{5}}\delta^3_{01}$$
$$\dot{\delta}^3_{01} = \frac{77\sqrt{5}}{5\sqrt{7}}\delta^3_{00} - 5\delta^3_{01} - \frac{12\sqrt{5}}{5\sqrt{3}}\delta^3_{02} \quad , 0 \le t \le \frac{1}{2}$$
$$\dot{\delta}^3_{02} =- \frac{14\sqrt{3}}{\sqrt{7}}\delta^3_{00} + \frac{14\sqrt{3}}{\sqrt{5}}\delta^3_{01} - 3\delta^3_{02} - 5\sqrt{3}\delta^3_{03}$$
$$\dot{\delta}^3_{03} = \frac{14}{\sqrt{7}}\delta^3_{00} - \frac{10}{\sqrt{5}}\delta^3_{01} + \frac{21}{\sqrt{3}}\delta^3_{02} + 15\delta^2_{03}$$

and

$$\dot{\delta}^3_{10} =- 7\delta^3_{10} - \frac{\sqrt{7}}{\sqrt{5}}\delta^3_{11}$$
$$\dot{\delta}^3_{11} = \frac{77\sqrt{5}}{5\sqrt{7}}\delta^3_{10} - 5\delta^3_{11} - \frac{12\sqrt{5}}{5\sqrt{3}}\delta^3_{12} \quad , 0 \le t \le \frac{1}{2}$$
$$\dot{\delta}^3_{12} =- \frac{14\sqrt{3}}{\sqrt{7}}\delta^3_{10} + \frac{14\sqrt{3}}{\sqrt{5}}\delta^3_{11} - 3\delta^3_{12} - 5\sqrt{3}\delta^3_{13}$$
$$\dot{\delta}^3_{13} = \frac{14}{\sqrt{7}}\delta^3_{10} - \frac{10}{\sqrt{5}}\delta^3_{11} + \frac{21}{\sqrt{3}}\delta^3_{12} + 15\delta^2_{13}$$

One can write the above equations as

$\dot{\delta}^3(t) = D\delta^3(t)$, where

$$D1 = \begin{pmatrix} -7 & \frac{\sqrt{7}}{\sqrt{5}} & 0 & 0 \\ \frac{77\sqrt{5}}{2\sqrt{7}} & -5 & -\frac{12\sqrt{5}}{5\sqrt{3}} & 0 \\ -\frac{14\sqrt{3}}{\sqrt{7}} & \frac{14\sqrt{3}}{\sqrt{5}} & -3 & -5\sqrt{3} \\ \frac{14}{\sqrt{7}} & -\frac{10}{\sqrt{5}} & \frac{21}{\sqrt{3}} & 15 \end{pmatrix}$$

$m = 4$

$$\dot{\delta}^4_{00} =- 9\delta^4_{00} - \frac{3}{\sqrt{7}}\delta^4_{01}$$
$$\dot{\delta}^4_{01} = \frac{45\sqrt{7}}{7}\delta^4_{00} - 7\delta^4_{01} - \frac{16\sqrt{7}}{7\sqrt{5}}\delta^4_{02}$$
$$\dot{\delta}^4_{02} =- 6\sqrt{5}\delta^4_{00} + \frac{86\sqrt{5}}{5\sqrt{7}}\delta^4_{01} - 5\delta^4_{02} - \frac{21\sqrt{5}}{5\sqrt{3}}\delta^4_{03}, 0 \le t \le \frac{1}{2}$$

$\dot{\delta}_{04}^4 = -\frac{786}{29}\delta_{00}^4 - \frac{16618}{145\sqrt{7}}\delta_{01}^4 - \frac{3494}{29\sqrt{5}}\delta_{02}^4 - \frac{5352}{145\sqrt{3}}\delta_{03}^4 + \frac{66}{29}\delta_{04}^4$

and

$$\dot{\delta}_{10}^4 = -9\delta_{10}^4 - \frac{3}{\sqrt{7}}\delta_{11}^4$$

$$\dot{\delta}_{11}^4 = \frac{45\sqrt{7}}{7}\delta_{10}^4 - 7\delta_{11}^4 - \frac{16\sqrt{7}}{7\sqrt{5}}\delta_{12}^4$$

$$\dot{\delta}_{12}^4 = -6\sqrt{5}\delta_{10}^4 + \frac{86\sqrt{5}}{5\sqrt{7}}\delta_{11}^4 - 5\delta_{12}^4 - \frac{21\sqrt{5}}{5\sqrt{3}}\delta_{13}^4$$

$0 \le t \le \frac{1}{2}$

$\dot{\delta}_{14}^4 = -\frac{786}{29}\delta_{10}^4 - \frac{16618}{145\sqrt{7}}\delta_{11}^4 - \frac{3494}{29\sqrt{5}}\delta_{12}^4 - \frac{5352}{145\sqrt{3}}\delta_{13}^4 + \frac{66}{29}\delta_{14}^4$

One can write the above equations as $\dot{\delta}^4(t) = D\delta^4(t)$, where

$$\begin{pmatrix} -9 & -\frac{3}{\sqrt{7}} & 0 & 0 & 0 \\ \frac{45\sqrt{7}}{7} & -7 & -\frac{16\sqrt{7}}{7\sqrt{5}} & 0 & 0 \\ -6\sqrt{5} & \frac{86\sqrt{5}}{5\sqrt{7}} & -5 & -\frac{21\sqrt{5}}{5\sqrt{3}} & 0 \\ -\frac{786}{29} & -\frac{16618}{145\sqrt{7}} & -\frac{3494}{29\sqrt{5}} & -\frac{5352}{145\sqrt{3}} & \frac{66}{29} \end{pmatrix}$$

5. Powers in terms of (BSWFs)

The power of t can be rewritten in terms of BSWFs as follows

$$A = T \times B$$

when $m = 1$

$$T = \begin{pmatrix} \frac{3}{2\sqrt{6}} & \frac{1}{2\sqrt{2}} & 0 & 0 \\ \frac{1}{4\sqrt{6}} & \frac{1}{4\sqrt{2}} & 0 & 0 \end{pmatrix} \quad \text{where } 0 \le t \le \frac{1}{2}$$

$$T = \begin{pmatrix} 0 & 0 & \frac{3}{2\sqrt{6}} & \frac{1}{2\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{2}} \end{pmatrix} \quad \text{where } \frac{1}{2} \le t \le 1$$

where $A = (1 \quad t)^T$ and $B = (\delta_{00}^1 \ \delta_{01}^1 \ \delta_{10}^1 \ \delta_{11}^1)^T$

when $m = 2$

$$\begin{pmatrix} \frac{5}{3\sqrt{10}} & \frac{1}{\sqrt{6}} & \frac{1}{3\sqrt{2}} & 0 & 0 & 0 \\ \frac{5}{24\sqrt{10}} & \frac{3}{8\sqrt{6}} & \frac{1}{6\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{24\sqrt{10}} & \frac{1}{8\sqrt{6}} & \frac{1}{12\sqrt{2}} & 0 & 0 & 0 \end{pmatrix}$$

where $0 \le t \le \frac{1}{2}$

$$\begin{pmatrix} 0 & 0 & 0 & \frac{2}{3\sqrt{10}} & \frac{3}{4\sqrt{6}} & \frac{1}{3\sqrt{2}} \\ 0 & 0 & 0 & \frac{25}{24\sqrt{10}} & \frac{7}{8\sqrt{6}} & \frac{1}{3\sqrt{2}} \\ 0 & 0 & 0 & \frac{5}{3\sqrt{10}} & \frac{1}{\sqrt{6}} & \frac{1}{3\sqrt{2}} \end{pmatrix}$$

where $\frac{1}{2} \le t \le 1$

where $A = (1 \quad t \quad t^2)^T$ and $B = (\delta_{00}^2 \ \delta_{01}^2 \ \delta_{02}^2 \ \delta_{10}^2 \ \delta_{11}^2 \ \delta_{12}^2)^T$

when $m = 3$

$$\begin{pmatrix} \frac{7}{4\sqrt{14}} & \frac{5}{4\sqrt{10}} & \frac{3}{4\sqrt{6}} & \frac{1}{4\sqrt{2}} & 0 & 0 & 0 & 0 \\ \frac{7}{40\sqrt{14}} & \frac{3}{8\sqrt{10}} & \frac{13}{40\sqrt{6}} & \frac{1}{8\sqrt{2}} & 0 & 0 & 0 & 0 \\ \frac{7}{240\sqrt{14}} & \frac{5}{48\sqrt{10}} & \frac{11}{80\sqrt{6}} & \frac{1}{16\sqrt{2}} & 0 & 0 & 0 & 0 \\ \frac{1}{160\sqrt{14}} & \frac{1}{32\sqrt{10}} & \frac{9}{160\sqrt{6}} & \frac{1}{32\sqrt{2}} & 0 & 0 & 0 & 0 \end{pmatrix}$$

where $0 \le t \le \frac{1}{2}$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \frac{7}{4\sqrt{14}} & \frac{5}{4\sqrt{10}} & \frac{3}{4\sqrt{6}} & \frac{1}{4\sqrt{2}} \\ 0 & 0 & 0 & 0 & \frac{21}{20\sqrt{14}} & \frac{1}{\sqrt{10}} & \frac{7}{10\sqrt{6}} & \frac{1}{4\sqrt{2}} \\ 0 & 0 & 0 & 0 & \frac{77}{120\sqrt{14}} & \frac{19}{24\sqrt{10}} & \frac{13}{20\sqrt{6}} & \frac{1}{4\sqrt{2}} \\ 0 & 0 & 0 & 0 & \frac{2}{5\sqrt{14}} & \frac{5}{8\sqrt{10}} & \frac{3}{5\sqrt{6}} & \frac{1}{4\sqrt{2}} \end{pmatrix}$$

where $\frac{1}{2} \le t \le 1$

where $A = (1 \quad t \quad t^2 \quad t^3)^T$

and $B = (\delta_{00}^3 \ \delta_{01}^3 \ \delta_{02}^3 \ \delta_{03}^3 \ \delta_{10}^3 \ \delta_{11}^3 \ \delta_{12}^3 \ \delta_{13}^3)^T$

6. Numerical Examples

The application problems in this work are

Example 1

This example clarifies the following concepts

Find the optimal state and optimal control based on

minimizing the performance index $= \int_0^1 \left(x(t) - \frac{1}{2}u(t)^2\right)dt,\ 0 \le t \le 1$
subject to $u(t) = \dot{x}(t) + x(t)$ with the condition $x(0) = 0, x(1) = \frac{1}{2}(1-\frac{1}{e})^2$

The exact solution for the state $x(t)$ and the control $u(t)$ is

$x(t) = 1 - 0.5e^{t-1} - 0.8160603e^{-t}$
$u(t) = 1 - e^{t-1}$
and $J_{exact} = 0.08404562020$

Example 2

Consider the linear control system, which consists of minimizing $J = \frac{1}{2}\int_0^1 (3x(t)^2 + u(t)^2)dt$
subject to $u(t) = \dot{x}(t) + x(t)$, $x(0) = 0$, $x(1) = 2$ and $J_{exact} = 6.1586$.

Example 1 is solved using BSWFs as follows

Let the initial approximation of x(t) is

$X^1(t) = x^0 + (x^1 - x^0)\left(\frac{1}{4\sqrt{6}}\delta^1_{00} + \frac{1}{4\sqrt{2}}\delta^1_{01} + \frac{1}{\sqrt{6}}\delta^1_{10} + \frac{1}{2\sqrt{6}}\delta^1_{11}\right)$

That is,

$x^1(t) = \frac{0.199788}{\sqrt{6}}\delta^1_{10} + \frac{0.099894}{\sqrt{6}}\delta^1_{11}$ or $x^1(t) = d_1\delta^1(t)$

$\dot{x}^1(t) = 0.049947\sqrt{6}\,\delta^1_{10} + \frac{0.09989}{\sqrt{2}}\delta^1_{11}$ or $\dot{x}^1(t) = r_1\dot{\delta}^1(t)$

$u^1(t) = 0.083245\sqrt{6}\,\delta^1_{10} + \frac{0.199788}{\sqrt{2}}\delta^1_{11}$ or $u^1(t) = w_1\left(\delta^1(t) + \dot{\delta}^1(t)\right)$

where $d_1 = [0 \quad 0 \quad \frac{0.199788}{\sqrt{6}} \quad \frac{0.099894}{\sqrt{2}}]^T$,

$r_1 = [0 \quad 0 \quad 0.049947\sqrt{6} \quad \frac{0.09989}{\sqrt{2}}]^T$

$w_1 = [0 \quad 0 \quad 0.083245\sqrt{6} \quad \frac{0.199788}{\sqrt{2}}]^T$

and $\delta^1(t) = [\delta^1_{00} \quad \delta^1_{01} \quad \delta^1_{10} \quad \delta^1_{11}]$
$\dot{\delta}^1(t) = [\dot{\delta}^1_{00} \quad \dot{\delta}^1_{01} \quad \dot{\delta}^1_{10} \quad \dot{\delta}^1_{11}]^T$

Now the second approximation of x and u

$$x^2 = d_1\delta^1 + a_2\,(\delta^2 - \delta^1)$$

where a_2 is the unknown parameter, here $a_2 = -0.409139$, then $x^2(t)$ and $u^2(t)$ can be written as $x^2(t) = d_1\delta^1(t) + d_2\delta^2(t)$ where

and $u^2 = w_1\left(\delta^1 + \dot{\delta}^1\right) + w_2\left(\delta^2 + \dot{\delta}^2\right)$
where

$$d_2 = \left[\frac{0.06818983}{\sqrt{6}} \quad \frac{0.10228475}{\sqrt{4}} \quad \frac{0.03409492}{\sqrt{2}} \quad \frac{0.15342713}{\sqrt{10}} \quad \frac{0.05114238}{2\sqrt{6}}\right]^T$$

$$w_1 = \left[0 \quad 0 \quad 0.083245\sqrt{6} \quad \frac{0.199788}{\sqrt{2}}\right]^T$$

$$w_2 = \left[\frac{0.57961358}{\sqrt{10}} \quad \frac{0.2045695}{\sqrt{6}} \quad -\frac{0.01704749}{\sqrt{10}} \quad \frac{0.03409492}{\sqrt{2}} \quad -\frac{0.25571188}{\sqrt{6}} \quad -\frac{0.13637967}{\sqrt{2}}\right]^T$$

Similarly, the third approximation for $x(t)$ and $u(t)$ can be found

$$x^3(t) = x^2(t) + a_3\,(\delta^3(t) - \delta^2(t))$$

Here a_3 =0.011654 and $x^3(t) = d_1\delta^1(t) + d_2\delta^2(t) + d_3\delta^3(t)$

where

$$d_3 = \left[-\frac{0.00026707}{\sqrt{6}} \quad -\frac{0.00084977}{\sqrt{10}} \quad -\frac{0.00094689}{\sqrt{6}} \quad -\frac{0.00036419}{\sqrt{2}} \quad -\frac{0.00281638}{\sqrt{14}} - \frac{0.00194233}{\sqrt{10}} \quad -\frac{0.0005827}{\sqrt{6}}\right]^T$$

$$u^2 = w_1\left(\delta^1 + \dot{\delta}^1\right) + w_2\left(\delta^2 + \dot{\delta}^2\right) + w_3\left(\delta^3 + \dot{\delta}^3\right)$$

where

$$u_3 = \left[-\frac{0.00332625}{\sqrt{6}} \quad -\frac{0.00594831}{\sqrt{10}} \quad \frac{0.02410921}{\sqrt{6}} \quad -\frac{0.00109256}{\sqrt{2}} \quad -\frac{0.00485583}{\sqrt{14}} \quad \frac{0.00242792}{\sqrt{10}} \quad -\frac{0.005827}{\sqrt{6}}\right]^T$$

The value of J are shown in **Table 1.**

Iteration	Our method	Error
1	0.08401526011	3.036×10-5
2	0.08402496173	2.065×10-5
3	0.08402530645	2.031×10-5

Table 1. The results of J for example 1 using BSWFs

Example 2 is solved using BSWFs as follows

Let the initial approximation of $x(t)$ is

$X^1(t) = x^0 + (x^1 - x^0)\left(\frac{1}{4\sqrt{6}}\delta^1_{00} + \frac{1}{4\sqrt{2}}\delta^1_{01} + \frac{1}{\sqrt{6}}\delta^1_{10} + \frac{1}{2\sqrt{6}}\delta^1_{11}\right)$

Therefore;

$x^1(t) = \frac{1}{\sqrt{6}}\delta^1_{10} + \frac{1}{2\sqrt{6}}\delta^1_{11}$ or $x^1 = d_1\delta^1$

$\dot{x}^1 = \frac{1}{\sqrt{6}}\delta^1_{10} + \frac{1}{2\sqrt{6}}\delta^1_{11}$ or $\dot{x}^1 = r_1\dot{\delta}^1$

$u^1 = \frac{5\sqrt{6}}{6}\delta^1_{10} + \frac{2}{\sqrt{2}}\delta^1_{11}$ or $u^1(t) = w_1\left(\delta^1(t) + \dot{\delta}^1(t)\right)$

where $d_1 = [0 \quad 0 \quad \frac{2}{\sqrt{6}} \quad \frac{1}{\sqrt{2}}]^T$, $r_1 = [0 \quad 0 \quad \frac{\sqrt{6}}{2} \quad \frac{1}{\sqrt{2}}]^T$, $w_1 = [0 \quad 0 \quad \frac{5\sqrt{6}}{6} \quad \frac{2}{\sqrt{2}}]^T$

and $\delta^1(t) = [\delta^1_{00} \quad \delta^1_{01} \quad \delta^1_{10} \quad \delta^1_{11}]$, $\dot{\delta}^1(t) =$

$[\dot{\delta}_{00}^{1} \quad \dot{\delta}_{01}^{1} \quad \dot{\delta}_{10}^{1} \quad \dot{\delta}_{11}^{1}]^{T}$

Now the second approximation of x and u

$$x^2 = d_1\delta^1 + a_2\,(\delta^2 - \delta^1)$$

where a_2 is the unknown parameter, here $a_2 = 1.4286$, then $x^2(t)$ and $u^2(t)$ can be written as

$$x^2(t) = d_1\delta^1(\mathrm{t}) + d_2\delta^2(t)$$

where

$$d_2 = \left[-\frac{2381}{10000\sqrt{10}} \quad -\frac{7143}{20000\sqrt{6}} \quad -\frac{2381}{20000\sqrt{2}} \quad -\frac{0.535725}{\sqrt{10}} \quad -\frac{7143}{20000\sqrt{6}}\right]^T$$

and the control variable can be evaluated to be

$$u^2(t) = w_1\left(\delta^1 + \dot{\delta}^1\right) + w_2\left(\delta^2 + \dot{\delta}^2\right)$$

where

$$w_1 = \left[0 \quad 0 \quad \frac{5\sqrt{6}}{6} \quad \frac{2}{\sqrt{2}}\right]^T$$

$$w_2 = \left[\frac{2.02385}{\sqrt{10}} \quad -\frac{7143}{10000\sqrt{6}} \quad +\frac{2381}{40000\sqrt{2}} \quad -\frac{2381}{20000\sqrt{10}} \quad \frac{7143}{8000\sqrt{6}} \quad \frac{2381}{50000\sqrt{2}}\right]^T$$

Similarly, the third approximation for $x(t)$ and $u(t)$ can be found

$$x^3(t) = x^2(t) + a_3\,(\delta^3(t) - \delta^2(t))$$

Here $a_3 = 0.3888$ and $x^3(t) = d_1\delta^1(t) + d_2\delta^2(t) + d_3\delta^3(t)$

where

$$d_3 = \left[-\frac{891}{10000\sqrt{14}} \quad -\frac{567}{20000\sqrt{10}} \quad -\frac{0.03159}{\sqrt{6}} \quad -\frac{243}{20000\sqrt{2}} \quad -\frac{2349}{25000\sqrt{14}} \quad -\frac{81}{1250\sqrt{10}} \quad -\frac{243}{12500\sqrt{6}}\right]^T$$

$$u^2(t) = w_1\left(\delta^1 + \dot{\delta}^1\right) + w_2\left(\delta^2 + \dot{\delta}^2\right) + w_3\left(\delta^3 + \dot{\delta}^3\right)$$

where

$$w_3 = \left[-\frac{0.11097}{\sqrt{14}} \quad -\frac{3969}{20000\sqrt{10}} \quad +\frac{0.80433}{\sqrt{6}} \quad -\frac{729}{20000\sqrt{2}} \quad -\frac{81}{500\sqrt{14}} \quad \mp\frac{81}{1000\sqrt{10}} \quad -\frac{243}{1250\sqrt{6}}\right]^T$$

For optimal values of the performance index J corresponding to $n = 1$, $n = 2$, $n = 3$ that is when $x(t) = x^1$, $u(t) = u^1$, $x(t) = x^2$, $u = u^2$, $x = x^3$, $u = u^3$ respectively, one refers to Table 2. Also, the value of J is shown in **Table 2.**

Iteration	Our method	Error
1	6.19047619	0.0319
2	6.177513228	0.0189
3	6.174827155	0.0167

Table 2. The results of cost functional J of example 2 using BSWFs.

Note the exact value of J is $J = 6.1586$.

7. Discussion

The solution of optimal control problems were obtained using basic spline wavelets based on operational matrix of derivative. The algorithm is comparable in terms of accuracy depending on the exact errors if the exact value of the performance index is known.

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