ORIGINAL ARTICLE

Operational Matrices of Derivative and Product for Shifted Chebyshev Polynomials of Type Three

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Abstract: In this paper an explicit expression for constructing operational matrices of derivative and product based on shifted chebyshev polynomials of type three are first presented. Then the conversion of power form basis to shifted chebyshev polynomials of the type three is listed through this work.

Keywords: Operational matrix of derivative; Operational matrix of product; Shifted chebyshev polynomials of type three

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1. Introduction

There are many types of Chebyshev polynomials, in particular the first and second Chebyshev polynomials as well as third and fourth Chebyshev polynomials[1-3]. The Chebyshev polynomials are useful and suitable in numerical analysis, such as integral equation, polynomial approximation, and spectral methods for differential equation, partial differential equations as well as optimal control problems and mathematical model[4-8]. Chebyshev polynomials are the eigenfunctions of the problem for a singular sturm-Liouville and they have various advantages, their expansion coefficients convergence are faster than other polynomials. They are widely utilized in numerical computation. One of the most important property is the operational matrix of derivative for polynomials and had been determined for B-spline polynomials[9], generalized Laguerre polynomials[10], Chebyshev polynomials[1], Boubaker polynomials[12-13], normalized Boubaker polynomials[14], shifted fourth Chebyshev wavelets, and orthonormal Bernstein polynomials[16-17].

In this paper, new explicit expressions concerning shifted Chebyshev polynomials of type three-named operational matrix of derivative and product operational matrix for are first derived then some other important properties are presented through this work. The paper is organized as follows: the shifted Chebyshev polynomials of type three is defined in section 2, while an explicit expression for operational matrix of derivative for SCTT is presented in section 3, another important matrix named the product operational matrix for SCTT is derived in section 4 with an exact formula to compute such matrix. The relation between the power and SCTT is included in section 5. In section 6, the discussion is listed.

2. Properties of Shifted Chebyshev Polynomials of Type Three

Shifted Chebyshev polynomials of the third kind $\overline{V}_n(\tau)$ are defined by

$$\overline{V_0}(\tau) = 1$$
$$\overline{V_1}(\tau) = 4\tau - 3$$

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with the recursive formula given by

$$\overline{V}_{n+1}(\tau) = 2(2\tau - 1)\overline{V}_n(\tau) - \overline{V}_{n-1}(\tau) \quad (1)$$

where n=1, 2, … and $0 \le \tau \le 1$

The orthogonal property of $\overline{V}_n(\tau)$ on [0, 1] are given by

$$\int_0^1 \overline{V}_m(\tau)\,\overline{V}_n(\tau)\,w(\tau)\,d\tau = \begin{cases} \frac{\pi}{2} & n = m \\ 0 & n \ne m \end{cases} \quad (2)$$

where $w(\tau)$ is the weight function and equal to $w(\tau) = \sqrt{\frac{\tau}{1-\tau}}, \tau \ne 1$

The coefficients of τ^k *in* $\overline{V}_n(\tau)$ are listed in **Table 1**.

k \ τ	0	1	2	3	4	5
0	1	-3	5	-7	9	-11
1		4	-20	56	-120	220
2			16	-112	432	-1232
3				64	-576	2816
4					256	-2516
5						1625

Table 1. Coefficients of τ^k *in* $\overline{V}_n(\tau)$

3. Operational Matrix of Derivative for SCTT

The operational matrix of derivative for $\overline{V}_n(\tau)$ will be presented throughout this section. To illustrate the formulation of the derivative matrix $n = 6$ is selected.

$\overline{V}_0(\tau) = 1$

$\overline{V}_1(\tau) = 4\tau - 3$

$\overline{V}_2(\tau) = 16\tau^2 - 20\tau + 5$

$\overline{V}_3(\tau) = 64\tau^3 - 112\tau^2 + 56\tau - 7$

$\overline{V}_4(\tau) = 256\tau^4 - 576\tau^3 + 432\tau^2 - 120\tau + 9$

$\overline{V}_5(\tau) = 1024\tau^5 - 2816\tau^4 + 2816\tau^3 - 1232\tau^2 + 220\tau - 11$

$\overline{V}_6(\tau) = 4096\tau^6 - 13312\tau^5 + 16640\tau^4 - 9984\tau^3 + 2912\tau^2 - 364\tau + 13$

One can get the derivative of the above polynomials in terms of $\overline{V}_n(\tau)$ to be,

$\dot{\overline{V}}_0(\tau) = 0$

$\dot{\overline{V}}_1(\tau) = 4\overline{V}_0(\tau)$

$\dot{\overline{V}}_2(\tau) = 4\,\overline{V}_0(\tau) + 8\,\overline{V}_1(\tau)$

$\dot{\overline{V}}_3(\tau) = 8\,\overline{V}_0(\tau) + 4\,\overline{V}_1(\tau) + 12\,\overline{V}_2(\tau)$

$\dot{\overline{V}}_4(\tau) = 8\,\overline{V}_0(\tau) + 12\,\overline{V}_1(\tau) + 4\,\overline{V}_2(\tau) + 16\,\overline{V}_3(\tau)$

$\dot{\overline{V}}_5(\tau) = 12\,\overline{V}_0(\tau) + 8\,\overline{V}_1(\tau) + 16\,\overline{V}_2(\tau) + 4\,\overline{V}_3(\tau) + 20\,\overline{V}_4(\tau)$

$\dot{\overline{V}}_6(\tau) = 12\,\overline{V}_0(\tau) + 16\,\overline{V}_1(\tau) + 8\,\overline{V}_2(\tau) + 20\,\overline{V}_3(\tau) + 4\,\overline{V}_4(\tau) + 24\,\overline{V}_5(\tau)$

The above equations can be written in matrix form as

$$\dot{\overline{V}}_n = D\overline{V}_n \quad (3)$$

where $\dot{\overline{V}}_n = [\dot{\overline{V}}_0 \;\; \dot{\overline{V}}_1 \dots \dot{\overline{V}}_6]^T$

$\overline{V}_n = [\,\overline{V}_0 \;\; \overline{V}_1 \dots \overline{V}_5\,]^T$

and the matrix D is an 7×6 matrix of derivative for $\overline{V}_n(\tau)$, the entries of D are:

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 2 & 1 & 3 & 0 & 0 & 0 \\ 2 & 3 & 1 & 4 & 0 & 0 \\ 3 & 2 & 4 & 1 & 5 & 0 \\ 3 & 4 & 2 & 5 & 1 & 6 \end{pmatrix} \quad (4)$$

In general, the entries of the matrix D can be constructed as

$$d_{ij} = \begin{cases} \frac{i+j}{2} & i + j \text{ even}, i > j \\ \left\lfloor \frac{i-j}{2} \right\rfloor + 1 & i + j \text{ odd}, i > j \end{cases} \quad (5)$$

and $d_{ij} = 0$ when $i < j$

Theorem 1

The derivative of SCTT is a linear combination of lower order SCTT by the following relation:

$$\dot{\overline{V}}_n(\tau) = 4\left[\sum_{i=1}^{n-1} \left(\left\lfloor \frac{n-j}{2} \right\rfloor + 1\right)_{i\ odd} \overline{V}_{i-1}(\tau) + \sum_{i=2}^{n} \frac{n+j}{2}_{\ i\ even} \overline{V}_{i-1}(\tau)\right] \quad (6)$$

For n even and

$$\dot{\overline{V}}_n(\tau) = 4\left[\sum_{i=1}^{n-1} \frac{n+j}{2}_{\ i\ odd} \overline{V}_{i-1}(\tau) + \sum_{i=2}^{n} \left(\left\lfloor \frac{n-j}{2} \right\rfloor + 1\right)_{i\ even} \overline{V}_{i-1}(\tau)\right] \quad (7)$$

for n odd.

4. Operational Matrix of Product for SCTT

It is important to evaluate the product of $\overline{V}_m(\tau)$ and $\overline{V}_n(\tau)$. In order to evaluate the product for the shifted chebyshev polynomials of type three, let $m = 3$, then

$$\overline{V}(\tau)\,\overline{V}^T(\tau)=\begin{pmatrix}\overline{V}_0\overline{V}_0 & \overline{V}_0\overline{V}_1 & \overline{V}_0\overline{V}_2 & \overline{V}_0\overline{V}_3\\ \overline{V}_1\overline{V}_0 & \overline{V}_1\overline{V}_1 & \overline{V}_1\overline{V}_2 & \overline{V}_1\overline{V}_3\\ \overline{V}_2\overline{V}_0 & \overline{V}_2\overline{V}_1 & \overline{V}_2\overline{V}_2 & \overline{V}_2\overline{V}_3\\ \overline{V}_3\overline{V}_0 & \overline{V}_3\overline{V}_1 & \overline{V}_3\overline{V}_2 & \overline{V}_3\overline{V}_3\end{pmatrix} \quad (8)$$

where

$\overline{V}_0\overline{V}_0=\overline{V}_0$

$\overline{V}_0\overline{V}_1=\overline{V}_1$

$\overline{V}_0\overline{V}_2=\overline{V}_2$

$\overline{V}_0\overline{V}_3=\overline{V}_3$

$\overline{V}_1\overline{V}_1=\overline{V}_2-\overline{V}_1+\overline{V}_0$

$\overline{V}_1\overline{V}_2=\overline{V}_3-\overline{V}_2+\overline{V}_1$

$\overline{V}_1\overline{V}_3=\overline{V}_4-\overline{V}_3+\overline{V}_2$

$\overline{V}_2\overline{V}_2=\overline{V}_4-\overline{V}_3+\overline{V}_2-\overline{V}_1+\overline{V}_0$

$\overline{V}_2\overline{V}_3=\overline{V}_5-\overline{V}_4+\overline{V}_3-\overline{V}_2+\overline{V}_1$

$\overline{V}_3\overline{V}_3=\overline{V}_6-\overline{V}_5+\overline{V}_4-\overline{V}_3+\overline{V}_2$

In general, the explicit formula to compute the product of $\overline{V}_m\,\overline{V}_n$ is listed as:

$$\overline{V}_m(\tau)\,\overline{V}_n(\tau)=\begin{cases}\sum_{k=i}^{j}(-1)^k\;\overline{V}_k(\tau) & if\ i\ even\\ \sum_{k=i}^{j}(-1)^{k+1}\;\overline{V}_k(\tau) & if\ i\ odd\end{cases}$$

(9)

where

$i=|m-n|$ and $j=m+n$

5. Conversion Between Power Basis and $\overline{V}_n(\tau)$

The power form representation of a polynomials are utilized in many systems, therefore; it is important to convert the power basis to shifted Chebyshev polynomials of type three.

The equivalent τ^i and $\overline{V}_i(\tau)$ forms are

$R(\tau)=\sum_{i=0}^{n}a_i\tau^i=\sum_{i=0}^{n}b_i\,\overline{V}_i(\tau)\ ,\quad \tau\epsilon[0,1]$

The polynomial $R(\tau)$ can be represented in the following two cases

$$R(\tau)=\sum_{i=0}^{n}a_i\tau^i=\begin{pmatrix}1 & \tau & \tau^2 & \dots & \tau^n\end{pmatrix}\begin{pmatrix}a_1\\ \vdots\end{pmatrix}=\tau A \quad (10)$$

or

$$R(\tau)=\sum_{i=0}^{n}b_i\overline{V}_i(\tau)=\begin{pmatrix}\overline{V}_0(\tau) & \overline{V}_1(\tau) & \dots & \overline{V}_n(\tau)\end{pmatrix}\begin{pmatrix}b_0\\ b_1\\ \vdots\\ b_n\end{pmatrix}=VB$$

that is

$$R(\tau)=\tau A=VB \quad (11)$$

For example, let $n=6$, one can obtain

$1=\overline{V}_0$

$\tau=\frac{1}{4}(\overline{V}_1+3\overline{V}_0)$

$\tau^2=\frac{1}{16}(\overline{V}_2+\ 5\overline{V}_1+10\,\overline{V}_0)$

$\tau^3=\frac{1}{64}(\overline{V}_3+7\,\overline{V}_2+\ 21\,\overline{V}_1+35\,\overline{V}_0)$

$\tau^4=\frac{1}{256}(\overline{V}_4+9\,\overline{V}_3+36\,\overline{V}_2+\ 84\,\overline{V}_1+126\,\overline{V}_0)$

$\tau^5=\frac{1}{256}(\overline{V}_5+11\overline{V}_4+55\,\overline{V}_3+165\,\overline{V}_2+\ 330\,\overline{V}_1+462\,\overline{V}_0)$

$\tau^6=\frac{1}{256}(\overline{V}_6+13\overline{V}_5+78\overline{V}_4+286\,\overline{V}_3+715\,\overline{V}_2+\ 1287\,\overline{V}_1+1716\,\overline{V}_0)$

In matrix form

$\tau=\mathrm{C}\,\overline{V}$

where $\tau=[1\quad \tau\quad \tau^2\quad \dots\quad \tau^6]^T$

$\overline{V}=[\overline{V}_0(\tau)\quad \overline{V}_1(\tau)\quad \overline{V}_2(\tau)\quad \dots\quad \overline{V}_6(\tau)]^T$

and c is given by

$$C=\frac{1}{2^{2i}}\begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0\\ 3 & 1 & 0 & 0 & 0 & 0 & 0\\ 10 & 5 & 1 & 0 & 0 & 0 & 0\\ 35 & 21 & 7 & 1 & 0 & 0 & 0\\ 126 & 84 & 36 & 9 & 1 & 0 & 0\\ 462 & 330 & 165 & 55 & 11 & 1 & 0\\ 1716 & 1287 & 715 & 286 & 78 & 13 & 1\end{pmatrix} \quad (12)$$

6. Conclusion

Some novel properties for shifted Chebyshev polynomials of type three are presented. They are operational matrices of derivative and product, the conversion of power form basis to shifted Chebyshev polynomials of the type three, which gives the relationship between them. All these properties can be applied to find the approximate solutions for many applications.

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