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ORIGINAL ARTICLE

Some Properties for Orthonormal Generalized B-spline Basis Polynomials

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Abstract: In this work, orthonormal generalized B-spline polynomials (OGBSPs) with some important properties are adopted. Their operational derivative matrix is first introduced. Then the relation for transformation of orthonormal generalized B-spline polynomials into B-spline polynomials is derived in this paper. In addition, the convergence is established which dictates that B-spline polynomials can converge to a smooth approximate solution.

Keywords: B-spline polynomials; Orthonormal polynomials; Operation matrix of derivative

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1. Introduction

Polynomials are the simplest tool in approximations. They are utilized to represent complicated functions and can be represented in many different bases for example, Chebyshev[1-2], Laguerre[3-4], B-spline[5-6], Bernstein[7-9], and other bases forms[10]. The orthonormal B-spline polynomials and their properties are important in many applications [11-16].

The B-spline polynomials and their basis form that can be generalized on the interval $[a, b]$, are defined as follows:

$$B_{k,m}(x) = \frac{1}{(b-a)^n}\binom{m}{k}(x-a)^k(b-x)^{m-k}\ ; \qquad k = 0,1,\ldots,m \tag{1}$$

For convenience, we set $B_{k,m}(x) = 0$, if $k < 0 \ or\ k > m.$

The useful properties for generalized B-spline polynomials are

1) The generalized B-spline polynomial of degree $\mathrm{m}-1$ interms of a linear combination of B-spline polynomials of degree m on the interval [a, b] is given as

$$(b-a)B_{k,m-1}(x) = \left(\frac{m-k}{m}\right)B_{k,m}(x) + \left(\frac{k+1}{m}\right)B_{k+1,m}(x)$$

2) The generalized B-spline polynomials of degree m can be represented by the combination of two B-spline polynomial of degree $m-1$

$$B_{k,m}(x) = \frac{1}{b-a}\left[(b-x)B_{k,m-1}(x) + (x-a)B_{k-1,m-1}(x)\right]$$

3) The derivatives of the m^{th} degree generalized B-spline polynomials are

$$\frac{d^n}{dx^n}\left(B_{k,m}(x)\right) = \frac{1}{(b-a)^n}\ \frac{m!}{(m-n)!}\sum_{h=max\,(0,k+n-m)}^{min\,(k,n)}(-1)^{h+n}\binom{n}{h}B_{k-h,m-n}(x)$$

4) The relation between generalized B-spline polynomials of degree m and the power basis is

$$\binom{m}{k}(x-a)^k(b-x)^{m-k} = \sum_{h=0}^{m-k}(-1)^h\binom{m}{k}\binom{m-k}{h}(x-a)^{h+k}$$

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2. Orthonormal Generalized B-spline Polynomials (OGBSPs)

An orthogonal sequence c for generalized B-spline polynomials can be generated over the interval $[a,b]$, with the aid of Gram-Schmidt orthonormalization process

To contract an orthogonal sequence φ_{i7} that spans the same subspace as the original set.

$$\varphi_{07} = B_{07}$$

$$\varphi_{k7} = B_{k7} - \sum_{j=1}^{k-1} c_{kj}\varphi_{j7}\ ,\ \ k = 1,2,...7$$

where

$$c_{kj} = (B_{k7}, \varphi_{j7}) / (\varphi_{j7}, \varphi_{j7})$$

and the orthogonal polynomials $\varphi_{k7}(x)$ can be normalized such that

$$O\varphi_{k7}(x) = \frac{\varphi_{k7}(x)}{||\varphi_{k7}||} = \frac{\varphi_{k7}(x)}{\sqrt{\int_a^b [\varphi_{k7}(x)]^2 dx}}$$

Therefore; the seventh generalized orthonormal B-spline Polynomials are

$$OB_{07} = \frac{\sqrt{15}}{\sqrt{b-a}}\frac{1}{(b-a)^7}[(b-t)^7]$$

$$OB_{17} = \frac{2\sqrt{13}}{\sqrt{b-a}}\frac{1}{(b-a)^7}[7(t-a)(b-t)^6 - \frac{1}{2}(b-t)^7]$$

$$OB_{27} = \frac{26\sqrt{11}}{7\sqrt{(b-a)}}\frac{1}{(b-a)^7}\Big[21(t-a)^2(b-t)^5 - 7(t-a)(b-t)^6 + \frac{7}{26}(b-t)^7\Big]$$

$$OB_{37} = \frac{123}{\sqrt{(b-a)}}\frac{1}{(b-a)^7}[35(t-a)^3(b-t)^4 - \frac{63}{2}(t-a)^2(b-t)^5 + \frac{63}{11}(t-a)(b-t)^6 - \frac{4}{77}(b-t)^7]$$

$$OB_{47} = \frac{66}{\sqrt{7(b-a)}}\frac{1}{(b-a)^7}[35(t-a)^4(b-t)^3 - 70(t-a)^3(b-t)^4 + 35(t-a)^2(b-t)^5 - \frac{14}{3}(t-a)(b-t)^6 + \frac{7}{66}(b-t)^7]$$

$$OB_{57} = \frac{12\sqrt{5}}{\sqrt{(b-a)}}\frac{1}{(b-a)^7}[21(t-a)^5(b-t)^2 - \frac{175}{2}(t-a)^4(b-t)^3 + 100(t-a)^3(b-t)^4 - \frac{75}{2}(t-a)^2(b-t)^5 + \frac{25}{6}(t-a)(b-t)^6 - \frac{1}{12}(b-t)^7]$$

$$OB_{67} = \frac{12\sqrt{3}}{\sqrt{(b-a)}}\frac{1}{(b-a)^7}[7(t-a)^6(b-t) - 63(t-a)^5(b-t)^2 + \frac{315}{2}(t-a)^4(b-t)^3 - 140(t-a)^3(b-t)^4 + 45(t-a)^2(b-t)^5 - \frac{9}{2}(t-a)(b-t)^6 + \frac{1}{12}(b-t)^7]$$

$$OB_{77} = \frac{8}{\sqrt{(b-a)}}\frac{1}{(b-a)^7}[(t-a)^7 - \frac{49}{2}(t-a)^6(b-t) + 147(t-a)^5(b-t)^2 - \frac{1225}{4}(t-a)^4(b-t)^3 + 245(t-a)^3(b-t)^4 - \frac{147}{2}(t-a)^2(b-t)^5 + 7(t-a)(b-t)^6 - \frac{1}{8}(b-t)^7]$$

3. The Relation Between OGBSP and GBSP

$$OB_{07} = \frac{\sqrt{15}}{\sqrt{b-a}}[B_{07}]$$

$$OB_{17} = \frac{2\sqrt{13}}{\sqrt{b-a}}[B_{17} - \frac{1}{2}B_{07}]$$

$$OB_{27} = \frac{26\sqrt{11}}{7\sqrt{b-a}}[B_{27} - B_{17} + \frac{7}{26}B_{07}]$$

$$OB_{37} = \frac{132}{\sqrt{b-a}}[B_{37} - \frac{3}{2}B_{27} + \frac{9}{11}B_{17} - \frac{7}{44}B_{07}]$$

$$OB_{47} = \frac{66}{\sqrt{7(b-a)}}[B_{47} - 2B_{37} + \frac{5}{3}B_{27} - \frac{2}{3}B_{17} + \frac{7}{66}B_{07}]$$

$$OB_{57} = \frac{12\sqrt{5}}{\sqrt{(b-a)}}[B_{57} - \frac{5}{2}B_{47} + \frac{100}{35}B_{37} - \frac{75}{42}B_{27} + \frac{25}{42}B_{17} - \frac{1}{12}B_{07}]$$

$$OB_{67} = \frac{12\sqrt{3}}{\sqrt{(b-a)}}[B_{67} - 3B_{57} + \frac{9}{2}B_{47} - 4B_{37} + \frac{25}{21}B_{27} - \frac{9}{14}B_{17} + \frac{1}{12}B_{07}]$$

$$OB_{77} = \frac{8}{\sqrt{(b-a)}}[B_{77} - \frac{7}{2}B_{67} + 7B_{57} - \frac{35}{4}B_{47} + 7B_{37} - \frac{7}{2}B_{27} + B_{17} - \frac{1}{8}B_{07}]$$

4. Fundamental Relation for Operational Matrix of Derivative For GBSP

On the interval $[a,b]$, any GBSP polynomials of degree m can be written as a linear combination of the GBSP basis polynomials of degree $m+1$

$$B_{k,m}(x) = \frac{m-k+1}{m+1}B_{k,m+1}(x) + \frac{k+1}{m+1}B_{k+1,m+1}(x) \qquad (2)$$

One can obtain the derivatives of nth-degree GBSP basis polynomials

$$\frac{d}{dx}B_{k,m}(x)=\frac{m}{b-a}[B_{k-1,m-1}(x)-B_{k,m-1}(x)$$

Furthermore, the first derivatives of m^{th} degree generalized Bernstein basis polynomials can be written as a linear combination of the generalized Bernstein basis polynomials of degree m

$$\dot{B}_{k,m}=\frac{1}{b-a}[(m-k+1)B_{k-1,m}(x)+(2k-m)B_{k,m}(x)-(k+1)B_{k+1,m}(x)]$$

There is a relation between GBSP basis polynomials matrix and their derivatives of the form

$B^{(l)}(x)-B(x)N^{l}$, $l=1,2,\ldots n$

Hence we obtain the matrix relation

$$N=\begin{bmatrix} \frac{-7}{b-a} & \frac{7}{b-a} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-1}{b-a} & \frac{-5}{b-a} & \frac{6}{b-a} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-2}{b-a} & \frac{-3}{b-a} & \frac{5}{b-a} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-3}{\mathrm{b}-a} & \frac{-1}{b-a} & \frac{4}{b-a} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-4}{b-a} & \frac{1}{b-a} & \frac{3}{b-a} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-5}{b-a} & \frac{3}{b-a} & \frac{2}{b-a} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-6}{b-a} & \frac{5}{b-a} & \frac{1}{b-a} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-7}{b-a} & \frac{7}{b-a} \end{bmatrix}$$

In other words. $\dot{B}(x)=B(x)N$

where

$$\dot{B}(x)=\left[\dot{B_{07}}(x),\dot{B_{17}}(x),\dot{B_{27}}(x),\dot{B_{37}}(x),\dot{B_{47}}(x),\dot{B_{57}}(x),\dot{B_{67}}(x),\dot{B_{77}}(x)\right]^{T}$$

$$B(x)=[B_{07}(x),B_{17}(x),B_{27}(x),B_{37}(x),B_{47}(x)B_{57}(x),B_{67}(x)B_{77}(x)]$$

5. The Derivative for OGBSP of Order Seven

$$O\dot{B}_{07}(x)=\frac{\sqrt{15}}{\sqrt{b-a}}[-7B_{07}(x)-B_{17}(x)]$$

$$O\dot{B}_{17}(x)=\frac{2\sqrt{13}}{\sqrt{b-a}}[\frac{21}{2}B_{07}(x)-\frac{9}{2}B_{17}(x)-2B_{27}(x)]$$

$$O\dot{B}_{27}(x)=\frac{26\sqrt{11}}{7\sqrt{b-a}}[-\frac{231}{26}B_{07}(x)+\frac{279}{26}B_{17}(x)-B_{27}(x)-3B_{37}(x)]$$

$$O\dot{B}_{37}(x)=\frac{132}{7\sqrt{b-a}}[\frac{301}{44}B_{07}(x)-\frac{569}{44}B_{17}(x)+\frac{173}{22}B_{27}(x)+\frac{7}{2}B_{37}(x)-4B_{47}(x)]$$

$$O\dot{B}_{47}(x)=\frac{66}{\sqrt{7(b-a)}}[-\frac{119}{22}B_{07}(x)+\frac{291}{22}B_{17}(x)-\frac{41}{3}B_{27}(x)+B_{37}(x)+9B_{47}(x)-5B_{57}(x)]$$

$$O\dot{B}_{57}(x)=\frac{12\sqrt{5}}{\sqrt{(b-a)}}[\frac{19}{4}B_{07}(x)-\frac{381}{28}B_{17}(x)+\frac{775}{42}B_{27}(x)-\frac{15}{2}B_{37}(x)-\frac{153}{14}B_{47}(x)+\frac{31}{2}B_{57}(x)-6B_{67}(x)]$$

$$O\dot{B}_{67}(x)=\frac{12\sqrt{3}}{\sqrt{(b-a)}}[-\frac{61}{12}B_{07}(x)+\frac{1343}{84}B_{17}(x)-\frac{176}{7}B_{27}(x)+\frac{109}{7}B_{37}(x)+\frac{23}{2}B_{47}(x)-\frac{59}{2}B_{57}(x)+23B_{67}(x)-7B_{77}(x)]$$

$$O\dot{B}_{77}(x)=\frac{8}{\sqrt{(b-a)}}[\frac{63}{8}B_{07}(x)-\frac{207}{8}B_{17}(x)+\frac{87}{2}B_{27}(x)-\frac{63}{2}B_{37}(x)-\frac{63}{4}B_{47}(x)+\frac{231}{4}B_{57}(x)+\frac{117}{2}B_{67}(x)+\frac{63}{2}B_{77}(x)-8B_{87}(x)]$$

6. Operational Matrix of Derivative for OGBSP

$$\begin{bmatrix} \frac{-27.110883}{\sqrt{b-a}} & \frac{-3.872983}{\sqrt{b-a}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{75.716577}{\sqrt{b-a}} & \frac{-32.449961}{\sqrt{b-a}} & \frac{-14.422205}{\sqrt{b-a}} & 0 & 0 & 0 & 0 & 0 \\ \frac{-109.448618}{\sqrt{b-a}} & \frac{132.191188}{\sqrt{b-a}} & \frac{-12.318892}{\sqrt{b-a}} & \frac{-36.956676}{\sqrt{b-a}} & 0 & 0 & 0 & 0 \\ 129 & \frac{-243.857143}{\sqrt{b-a}} & \frac{148.285714}{\sqrt{b-a}} & 66 & \frac{-75.428571}{\sqrt{b-a}} & 0 & 0 & 0 \\ \frac{-134.933317}{\sqrt{b-a}} & \frac{329.962985}{\sqrt{b-a}} & \frac{-340.923955}{\sqrt{b-a}} & \frac{24.945655}{\sqrt{b-a}} & \frac{224.510897}{\sqrt{b-a}} & \frac{-124.728276}{\sqrt{b-a}} & 0 & 0 \\ \frac{127.455875}{\sqrt{b-a}} & \frac{-365.117957}{\sqrt{b-a}} & \frac{495.129338}{\sqrt{b-a}} & \frac{-201.246118}{\sqrt{b-a}} & \frac{-293.244343}{\sqrt{b-a}} & \frac{415.908644}{\sqrt{b-a}} & \frac{-16.996894}{\sqrt{b-a}} & 0 \\ \frac{-105.655099}{\sqrt{b-a}} & \frac{332.306319}{\sqrt{b-a}} & \frac{-522.584472}{\sqrt{b-a}} & \frac{323.646065}{\sqrt{b-a}} & \frac{239.023011}{\sqrt{b-a}} & \frac{-613.145986}{\sqrt{b-a}} & \frac{478.046023}{\sqrt{b-a}} & \frac{-145.492268}{\sqrt{b-a}} \\ \frac{63}{\sqrt{b-a}} & \frac{-207}{\sqrt{b-a}} & \frac{348}{\sqrt{b-a}} & \frac{-252}{\sqrt{b-a}} & \frac{-126}{\sqrt{b-a}} & \frac{462}{\sqrt{b-a}} & \frac{-468}{\sqrt{b-a}} & \frac{252}{\sqrt{b-a}} \end{bmatrix}$$

7. Convergence criterion for OGBSPs

If the function $y(x)$ is expanded interns of OGBSP$y(x) = \sum_{k=0}^{\infty} y_k\, OB_k(x)$ (3)

It is not possible to perform computation an infinite number of terms; therefore, the series in Eq. 3 must be truncated. That is

$$y(x) = \sum_{k=0}^{m} y_k\, OB_k(x) \quad (4)$$

So that $y(x) = y_m(x) + \sum_{k=m+1}^{\infty} y_m\, OB_k(x)$ or $y(x) - y_m(x) = r(x)$

where

$$r(x) = \sum_{k=m+1}^{\infty} y_m\, OB_k(x) \quad (5)$$

The coefficients in and must be selected such that the norm of the residual function $||r(x)||$ is less than some convergence criteria ϵ, that is $||r(x)|| < \epsilon$

$$||r(x)||^2 = \int_a^b [\sum_{k=0}^{m+n} y_k\, OB_k(x) - \sum_{k=0}^{m} y_k\, OB_k(x)]^2 dx$$

$$= \int_a^b [\sum_{k=m+1}^{m+n} y_k\, OB_k(x)]^2 dx$$

$$= \int_a^b [\sum_{k=m+1}^{m+n} y_k\, OB_k(x)][\sum_{k=m+1}^{m+n} y_k\, OB_k(x)]\ dx$$

$$= \sum_{k=m+1}^{m+n} \sum_{h=m+1}^{m+n} y_k y_h \int_a^b OB_k(x) OB_h(x)\, dx$$

we have $\int_a^b OB_k(x) OB_h(x)\, dx = \begin{cases} 1 & if\ k = h \\ 0 & if\ k \neq h \end{cases}$

Then

$||r(x)||^2 = \sum_{k=m+1}^{m+n} y_k^2$ that is $\sum_{k=m+1}^{m+n} y_k^2 < \epsilon$

8. Discussion

The generalized orthonormal B-spline polynomials of order seven are first presented. Then some formulas that relate OGBSP with GBSP are obtained. Then, their operational derivative matrix is derived. In addition, the convergence is established, which dictates that generalized orthonormal B-spline polynomials can converge to a smooth approximate solution. The given results can be applied to solve optimal control problems and boundary value problems.

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