ORIGINAL ARTICLE

Three Step Homotopy Perturbation Iteration Algorithm for Nonlinear Equations

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Abstract: In this paper, an improved iterative three-step method with sixth order convergence based on Homotopy perturbation technique is suggested. It is named three step Homotopy Perturbation iteration algorithm (TSHPI). Four nonlinear test examples are solved with the proposed method and compared to other methods. The obtained results show that TSHPI method is a powerful tool and can generate highly accurate solutions with less iteration.

Keywords: Nonlinear equation; Homotopy perturbation method; Three-step algorithm

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1. Introduction

Several numerical and approximate methods have been proposed and analyzed with specific conditions to solve nonlinear equations $f(x) = 0$. These methods have been suggested based on various techniques such as[1-3]. In many areas of science and technology, the nonlinear problems can be solved with numerical methods. Some iterative algorithms have been suggested and analyzed in order to modify the order of convergence of Newton's method[4-6]. Some of the numerical methods are of two-step method or three-step method; many of them are depended on the second derivative of free of second derivative[7-9]. Different authors have proposed higher order with multi-step technique to solve real nonlinear equations, all with aim of increasing efficiency. In[10], three steps and fourth order is applied to solve nonlinear equation that model load flow in electric power systems while the authors in[11] used numerical algorithm based on Adomain decomposition method. An improved algorithm for solving Kepler's equation is presented in[12] for elliptical orbit. In Astrophysics[13], the author used the improving of predictor corrector Halley method. New predictor-corrector quadrature algorithms applied by[14] for solving Hyperbolic trajectory. Other applications methods can be found in[15-20]. Furthermore, homotopy method is another method used to solve nonlinear problems. Much attention has been presented to modify several iterative methods for treating nonlinear equations for example in[21] the Newton-homotopy method with start system is applied in Maple 4 to solve nonlinear equation. The authors in[22] applied higher order homotopy Taylor-perturbation for solving nonlinear equations. In this paper, we propose a modification to three step iteration method presented in[7] with using homotopy. Perturbation method to find roots of polynomials and the efficiency of the modified algorithm is illustrated by solving several examples.

2. Three Step Iteration Algorithm

In this section, a sixth order convergence with three steps is listed. For a nonlinear equation, let $f(x) = 0$, assume x_0 is initial solution, then the iteration solution x_{n+1} can be computed by the following scheme[7]

$$y_n = x_n - \frac{f(x_n)}{f^{'}(x_n)}.$$

$$z_n = x_n - [\frac{f(x_n)+f(y_n)}{f^{'}(x_n)}].$$

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$$x_{n+1} = z_n - \frac{f(z_n)f(x_n)}{f'(y_n)[f(x_n)+f(y_n)]-f'(x_n)f(y_n)} \qquad (1)$$

Lemma 1

Let f be the sufficiently smooth function in the real open domain D, and r is zero simple z of f Assume that x_0 is close to r, then Eq. 1 has six-order convergence and the error term satisfies

$$e_{n+1} = [c_2^3\left(188 + 8c_2^2 - 194\, c_3\right) + c_2^2 c_4(62c_2 - 62) + c_2^4(4 + 8c_2) + c_5(13c_2 - 13c_3)]e_n^6 + O(e_n^7)$$

where $c_j = \frac{f^{(j)}(r)}{j!\, f'(r)}$, $j = 2, 3, \dots$

proof: see ref.[7]

3. New TSHPI Algorithm

The illustration and construction of TSHPI method for nonlinear equations is discussed through this section.

Define the function $H(x,\lambda): \Re \times [0,1] \rightarrow \Re$ be the convex Homotopy function as

$$H(x,\lambda) = (1-\lambda)p(x) + q(x)\lambda = 0 \qquad (2)$$

where

- The values of: $0 < \lambda < 1$
- The start function system: $p(x)$
- The target system function: $q(x)$

and $H(x,0) = p(x)$, $H(x,1) = q(x) = f(x)$

By converting the equations in Eq. 1 to homotopy equations, one can get

$$y_n = x_n - \frac{H(x_n)}{H'(x_n)}.$$

$$z_n = x_n - [\frac{H(x_n)+H(y_n)}{H'(x_n)}].$$

$$x_{n+1} = z_n - \frac{H(z_n)H(x_n)}{H'(y_n)[H(x_n)+H(y_n)]-H'(x_n)H(y_n)} \quad n = 0,1,.. \qquad (3)$$

4. Start Function for TSHPI Method

In order to determine the starting value $x_\circ$ for the proposed algorithm given in Eq. 3, the following start system Newton homotopy will be used

$$p(x) = x^n - C \qquad (4)$$

where

- The highest power of x for $f(x)$: n
- The real number: C

Then the initial starting value $x_\circ$ can find easily as follows:

Let $p(x) = x^n - C = 0$

Note that the function $p(x)$ can be selected so that $p(x)$ has at least one trivial solution.

5. Application Results

Some numerical examples are considered in this section in order to illustrate the performance of the newly proposed method TSHPI. The following criteria is utilized for determining the root

$$\delta = |x_{n+1} - x_n| < \varepsilon$$

and the examples for comparison are listed in **Table 1**, while **Tables 2** to **5** show that the efficiency of TSHPI method as well as the iterations is less than δ in all computations converge.

Functions $f(x) = q(x)$	**$p(x)$**	**$x_\circ$: start point**
$f_1 = 1/2 + 1/4\, x^2 - \sin(x) - \cos(2x)$	$x^2 - 1/2$	0.7071
$f_2 = (x-2)^2(x^4 + 6x - 40)$	$x^4 - 40$	2.5148
$f_3 = (x-1)^3 - 1$	$x^3 - 1$	1
$f_4 = 5x^6 + 3x^4 - x^2 - 12$	$5x^6 - 12$	1.1571

Table 1. Table of functions

Value of λ	Roots	Iteration Number	$\lvert x_{n+1} - x_n \rvert$	Newton Method
0.1	0.7196	3	$1.8027e^{-09}$	
0.2	0.7317	3	$-1.4750e^{-10}$	
0.4	0.7546	3	$-1.1910e^{-10}$	3
0.6	0.7762	3	$-9.5034e^{-10}$	
0.8	0.7967	4	$-1.8733e^{-09}$	
0.9	0.8037	4	$1.8127e^{-09}$	

Table 2. Results for the function f_1 for different values to λ

Value of λ	Roots	Iteration Number	$\lvert x_{n+1} - x_n \rvert$	Newton Method
0.1	2.5082	3	$-1.7476e^{-07}$	
0.2	2.5008	3	$-4.7896e^{-08}$	
0.4	2.4828	3	$-1.4385e^{-08}$	5
0.6	2.45575	4	$-3.1003e^{-08}$	
0.8	2.4160	6	$-1.1213e^{-08}$	
0.9	2.3779	6	$-1.0209e^{-10}$	

Table 3. Results for the function f_2 for different values to λ

Value of λ	Roots	Iteration number	$\lvert x_{n+1} - x_n \rvert$	Newton method
0.1	0.7196	3	$1.8027e^{-09}$	
0.2	0.7317	3	$-1.4750e^{-10}$	
0.4	0.7546	3	$-1.1910e^{-10}$	3
0.6	0.7762	3	$-9.5034e^{-10}$	
0.8	0.7967	4	$-1.8733e^{-09}$	
0.9	0.8067	4	$1.8127e^{-09}$	

Table 4. Results for the function f_3 for different values to λ

Value of λ	Roots	Iteration number	$\lvert x_{n+1} - x_n \rvert$	Newton method
0.1	1.1507	3	$-7.8180e^{-06}$	
0.2	1.1444	3	$-2.1298e^{-09}$	
0.4	1.1323	3	$-1.3261e^{-07}$	4
0.6	1.1208	4	$-1.2225e^{-07}$	
0.8	1.1098	4	$8.8281e^{-09}$	
0.9	1.1045	4	$-1.7699e^{-09}$	

Table 5. Results for the function f_4 for different values to λ

6. Discussion

In order to assess the benefits of the proposed method TSHPI, some nonlinear problems are solved and compared with other algorithms. The results show that TSHPI provides highly accurate approximations with less iteration.

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