# Perturbation Analysis of ObSTP for Compressed Sensing

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**Abstract:** Many algorithms for compressed sensing are studied. And the common guarantee for the reconstruction algorithm is restricted isometry property (RIP), which is shown to only hold under ideal assumptions. However, in practice, more than one ideal condition is often violated and there is no RIP-based guarantee application. Based on this discrepancy, we propose a new oblique subspace thresholding pursuit (ObSTP) algorithm. It is guaranteed by the restricted biorthogonality property (RBOP) which requires no ideal assumptions. The ObSTP is an integration of the oblique pursuits and the subspace thresholding pursuit technique. The simulation results illustrate that the ObSTP algorithm has better performance.

**Keywords:** Compressed sensing, subspace thresholding pursuit, restricted isometry property, restricted biorthogonality property, perturbation

## I. Introduction

Compressed sensing (CS) technology aims at recovering a sparse signal from compressed measurements by finding the sparse solution to the underdetermined system y = Φx, i.e., solving the following $l_0$ problem:

$$\min \| \mathrm{x}\|_0 \quad \text{s.t.} \quad \mathrm{y}=\Phi \mathrm{x}, \tag{1}$$

where $\mathrm{y} \in \mathbb{R}^m$ denotes the observation vector, $\mathrm{x} \in \mathbb{R}^N$ is the signal vector and $\Phi \in \mathbb{R}^{m\times N}$ represents for the measurement matrix, with $m \ll N$, $\| \mathrm{x}\|_0=|\{i: x_i \neq 0\}|$ represents the $l_0$ -norm of x.

Candés, Tao and Donoho et al have shown that when the measurement matrix Φ satisfied the restricted isometry property (RIP), combinatorial optimization $l_0$ problem can be transformed into a convex optimization problem with $l_1$ constraints :

$$\min \| \mathrm{x}\|_1 \text{ s.t. } \mathrm{y}=\Phi \mathrm{x}. \tag{2}$$

In fact, the observation vector y is often contaminated by noise which we call it perturbation, and thus mode (2) was formulated as:

$$\min \| \mathrm{x}\|_1 \text{ s.t. } \mathrm{y}=\Phi \mathrm{x}+\mathrm{n}. \tag{3}$$

The common strategy solving the $l_1$ problem can be sorted into categories including convex optimization, heuristic algorithms, and thresholding algorithms. The convex optimization methods include $l_1$ minimization, , reweighted $l_1$ minimization, , and dual-density-based reweighted $l_1$ minimization, The heurisitic-type methods include orthogonal matching pursuit (OMP), and its variants such as the Regularized OMP, stagewise OMP, subspace pursuit (SP), and compressed sampling matching pursuit algorithms. The thresholding methods can be classified as soft thresholding, harding threholding, and optimal thresholding methods. In view of theoretical guarantees of greedy algorithms, the wellknown condition is the restricted isometry property (RIP)as follows:

Definition 1 (See definition 2 in ): For any $s$ -sparse signal $\mathrm{x} \in \mathbb{R}^N$ which satisfies with $\| \mathrm{x}\|_0 \le s$, the measurement matrix Φ satisfies the $s$ -order RIP if

$$(1-\delta_s) \| \mathrm{x} \|_2^2 \le \| \Phi \mathrm{x}\|_2^2 \le (1+\delta_s)\|\mathrm{x}\|_2^2, \tag{3}$$

where $0 \le \delta \le 1$. The infimum of $\delta$ denoted by $\delta_s$ is called the restricted isometry constant (RIC) of Φ.

The key assumption showing in the RIP is that the measurement matrix Φ satisfies the isotropy. That is, $\mathrm{E}\Phi^*\Phi = \mathrm{I}$, where E stands for the matrix expectation. However, in practice, the deviation measured by $\| \mathrm{E}\Phi^*\Phi - \mathrm{I} \|_2$ is not negligible, which is called the anisotropic case. In the case of anisotropic property, the authors in present the oblique pursuit method for compressed sensing. They introduced the oblique projection theory to the greedy algorithm and proposed the oblique matching pursuit (ObMP), oblique subspace pursuit (ObSP), oblique iterative hard thresholding (ObIHT), and oblique hard thresholding pursuit (ObHTP) algorithms. In these algorithms, the authors in put forward Oblique factor matrix $\tilde{\Phi}$ to satisfy that $\mathrm{E}\tilde{\Phi}^*\Phi = \mathrm{I}$ ,then used restricted biorthogonality property (RBOP) toanalyze the theoretical guarantees of the oblique pursuit algorithm. TheRBOP is defined as follows:

Definition 2 (See definition 1.9 in): The restricted biorthogonality constant $\theta_s(\tilde{\Phi}^*\Phi)$ of $\tilde{\Phi}^*\Phi \in \mathbb{R}^{N\times N}$ is defined as the smallest $\theta$ that satisfies

$$|\langle \mathrm{x}_1, \tilde{\Phi}^*\Phi \mathrm{x}_2\rangle - \langle \mathrm{x}_1, \mathrm{x}_2\rangle| \le \theta \| \mathrm{x}_1 \|_2 \| \mathrm{x}_2 \|_2,$$

for any two $s$ -sparse $\mathrm{x}_1$, $\mathrm{x}_2$ with common support. When $\tilde{\Phi} = \Phi$, the RBOPis equivalent to RIP.

The reference proposed a new greedy algorithm which is called STPalgorithm. This algorithm combines the SP algorithm and the HTP algorithm, which has strong recovery rate. In this paper, our main contributionis to propose the ObSTP algorithm which requires no ideal assumption onthe measurement matrix. The ObSTP algorithm has the same complexityas the STP algorithm and its performance is identical to the STP algorithm.Meanwhile, we use the RBOP to analyze theconvergence guarantees of theObSTP algorithm.

The rest of the paper is organized as follows. Section 2 introduces thepreliminaries for thispaper's main content. Section 3 presents the maintheorem of this paper. In section 4, we verify the performance of ObSTPalgorithm through simulation. The whole paper is concluded in

Section 5.Section 6 is the appendix which presents the detailed proof of the maintheorem.

## II.Preliminaries

We first define some notations that will be used in this paper. Let $\Gamma \subseteq \{1,2,\ldots,N\}$ and $|\Gamma|= s$ denotes the cardinality of $\Gamma$ . For $\mathrm{x} \in \mathbb{R}^N$ , $\mathrm{x}_\Gamma$ is the vector obtained from x that holds the $|\Gamma|$ entries in $\Gamma$ and sets all other entries to zero. The support of signal x is defined as supp(x). For any matrix $\Phi \in \mathbb{R}^{m\times N}$ , $\Phi^*$ denotes the transpose of $\Phi$ and $\Phi_\Gamma$ indicates the submatrix consisting of columns of $\Phi$ with indices in $S$ . For any vector $z \in \mathbb{R}^N$ , $H_\Gamma(z)$ denotes the operator that holds $|\Gamma|$ largest entries in vector $z$ and set other entries to zero.

To facilitate the following development, we first give the definition ofoblique pursuit in :

Definition 3 (See definition 2.3 in): Let $v_1, v_2 \subset H$ be two subspacessuch that $v_1 \oplus v_2{}^\perp = H$ . The oblique projection onto $v_1$ along $v_2{}^\perp$ ,denotedby $E_{v_1,v_2{}^\perp}$ , is defined as a linear map that satisfies

$$\begin{cases}(E_{v_1,v_2{}^\perp})\mathrm{x} = \mathrm{x},\text{if } \mathrm{x} \in v_1;\\ (E_{v_1,v_2{}^\perp})\mathrm{x} = 0,\text{if } \mathrm{x} \in v_2{}^\perp.\end{cases}$$

By the definition of oblique projection, it follows that $I_H$ - $E_{v_1,v_2{}^\perp} = E_{v_1{}^\perp,v_2}$ and $E_{v_1,v_2{}^\perp} = E_{v_2,v_1{}^\perp}$ where $I_H$ is the unit operator. For the two given matrix $\tilde{\Phi}$ and $\Phi$ whose columns forms bases for $v_1$ and $v_2$ respectively, it has that $E_{v_1,v_2{}^\perp} = \tilde{\Phi}(\Phi^*\tilde{\Phi})^{-1}\Phi^*$ .Then, we give a detailed description of the ObSTP algorithm in Algorithm 1.

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Algorithm 1: Oblique Subspace Thresholding Pursuit

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Inputs: $s, \mu, \Phi, \mathrm{y}$;

Initialization: $\Gamma^0$=$\varnothing$， $\mathrm{x}^0$=0.

Iteration:

At the $n$ -th iteration, go through following steps:

1) $\vartriangle\Gamma = \mathrm{supp}(H_S(\tilde{\Phi}^*\mathrm{y}_r^{n-1}))$;

2) $\tilde{\Gamma}^n = \Gamma^{n-1} \bigcup \vartriangle\Gamma$;

3) $\tilde{\mathrm{x}}^n = \arg\min_{z\in\mathbb{R}^N}\{\| \tilde{\Phi}^*(\mathrm{y}\text{-}\Phi \mathrm{z} \|_2 \quad \mathrm{z} \in \tilde{\Gamma}^n$;

4) $U^n = \mathrm{suup}(H_\Gamma(\tilde{\mathrm{x}}^n))$;

5) $\mathrm{u}^n = \mathrm{supp}(H_{U^n}(\tilde{\mathrm{x}}^n))$;

6) $\Gamma^n = \mathrm{supp}(H_\Gamma(\mathrm{u}^n + \mu\tilde{\Phi}^*(\mathrm{y}\text{-}\Phi \mathrm{u}^n)))$;

7) $\mathrm{x}^n = \arg\min_{z\in\mathbb{R}^N}\{\| \tilde{\Phi}^*(\mathrm{y}\text{-}\Phi \mathrm{z} \|_2 \quad \mathrm{z} \in \Gamma^n\}$.

Output:

1) $\mathrm{x}^n$， $\mathrm{supp}(\mathrm{x}^n)$ .

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In the ObSTP algorithm, the initial estimated signal is $\mathrm{x}^0$=0 ,and estimated supp(x) is $\Gamma^0$=$\varnothing$ .Before the execution of ObSTP, the parameters $\mu$ can be arbitrarily adjusted. In steps 1 and 7, the matrix $\tilde{\Phi}^*$ is used to identify the several largest entries. Usually, we set the matrix $\tilde{\Phi}$=$\Phi(\mathrm{E}\Phi^*\Phi)^{-1}$ .Different from the STP algorithm, ObSTP algorithm applies the obliquepursuit method to solve the least squares problem in steps 3 and 7.

## III.Main Results

This section provides the theoretical results about ObSTP algorithm when the measurement matrix $\Phi$ is anisotropic.

Theorem 1: For the general compressed sensing model in (6), define that supp(x) = $\Gamma$, $\theta = \theta_{3s}(\tilde{\Phi}^*\Phi)$ be the restricted biorthogonality constant of $\tilde{\Phi}^*\Phi$ and $\tilde{\delta}_{2S}$ be the RIC of $\tilde{\Phi}$ .If one of the following three cases: (1) $\frac{1+\theta}{2\theta\sqrt{1+2\theta}} < \mu < 1$; (2) $0 < \mu < \frac{1}{1+\theta} + \frac{1-\theta}{2\theta\sqrt{1+2\theta^2}}$; (3) $\theta_{3s} < 0.535$ occurs, then the sequence

$\mathrm{x}^n$ generated by the ObSTP algorithm satisfies

$\| \mathrm{x}_\Gamma\text{-}\mathrm{x}^n \|_2 \le \rho^n \| \mathrm{x}_\Gamma\|_2 + C\|\mathrm{n}_1\|_2$.

where $\rho = \frac{2\theta(|\mu\text{-}1|+\mu\theta)\sqrt{1+2\theta^2}}{1-\theta^2}$,

$$(1-\rho)C = \sqrt{\frac{1}{1-\theta^2}}[((\sqrt{2}+1)\theta t_1 + (2\sqrt{2}+1)t_2 + \sqrt{2(1+\theta)}\mu + \frac{\sqrt{1+\tilde{\delta}_{2s}}}{1-\theta}],$$

and $t_1 = \sqrt{\frac{1}{1-\theta^2}}\sqrt{2(1+\tilde{\delta}_{2s})} + \frac{\sqrt{1+\tilde{\delta}_{2s}}}{1-\theta}, t_2 = \sqrt{1+\tilde{\delta}_{2s}}$.

Proof: The proof is given in Appendix.

Remark 1: As shown in the appendix, $\rho < 1$ can be guaranteed. Then,the iterative sequence $x_n$ which generated from oblique STP algorithm is convergent.

IV.Numerical results

In the first, we use the ObSTP and STP algorithm to reconstruct the phantom image which has been transformed by wavelet transform. Simulations via synthetic data are carried out to demonstrate the numerical performance of the ObSTP, which is proposed in this paper. The compression ratio is 0.3, 0.4, 0.5, respectively. Phantom image recovered by ObSTP algorithm are then become clearer and take less time under the same compression rates, we obtain the ObSTP algorithm outperforms the STP algorithm in terms of recovery performance. The results of image recovery are shown in Figs. 1 and 2.

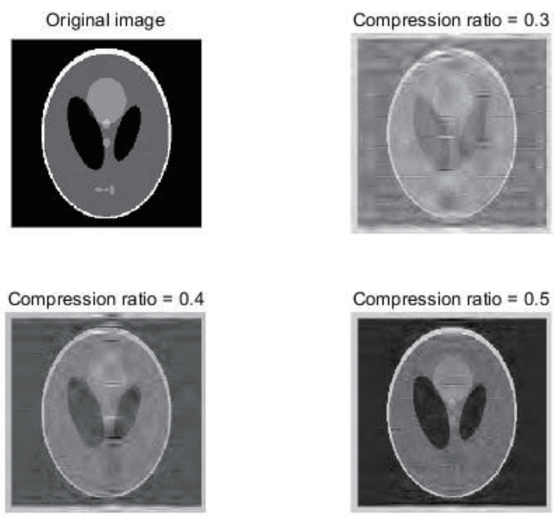


**Figure 1: Recovery performance of STP with different compression ratios**

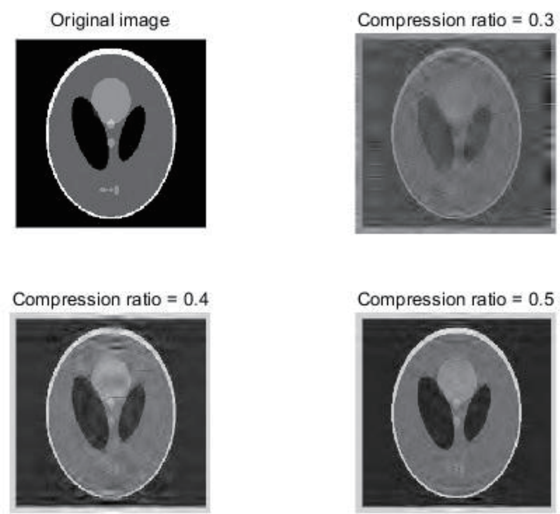


**Figure 2: Recovery performance of ObSTP with different compression ratios**

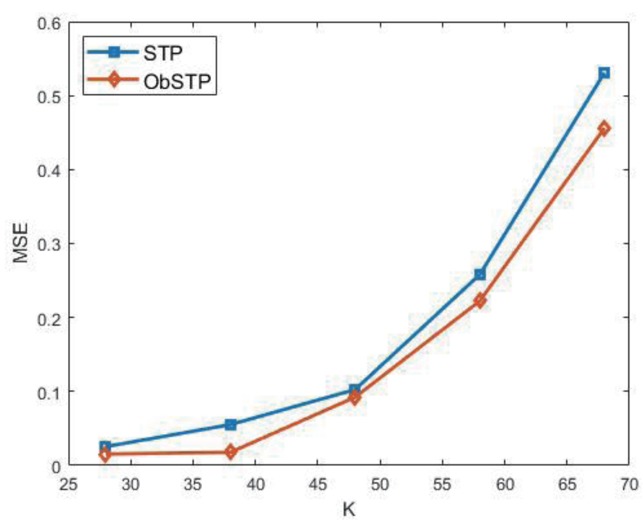


**Figure 3: Recovery performance of ObSTP and STP under different sparsity** $K$

In the second, we apply the ObSTP algorithm to the perturbation compressed sensing model (3) and use the mean square errors (MSE) to evaluate the algorithm's performance. The MSE is defined as $\text{MSE}=\frac{\|\hat{x}-x\|_2}{\|x\|_2}$, where $\hat{x}$ is the recovered signal and $x$ s the original signal. we set the parameters SNR = 20dB. Figs. 3 and 4. demonstrate that the MSE growth rate of the ObSTP algorithm is slower than that of the STP algorithm with the sparsity $K$ increase. This result indicates that ObSTP has stronger recovery performance than STP when there is the noise.

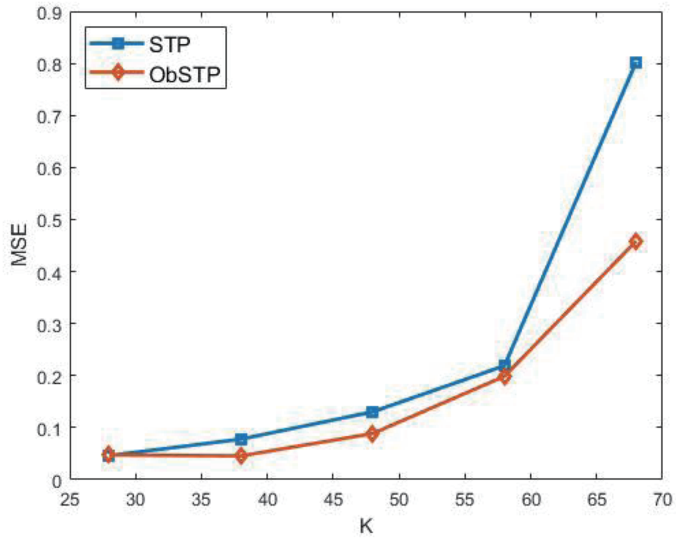


**Figure 4: Recovery performance of ObSTP and STP with perturbationunder different sparsity** $K$

## V. Conclusion

In this paper, we propose an ObSTP algorithm when the measurementmatrix is anisotropy. By the oblique pursuit method, we apply RBOP toanalyze the performance of ObSTP algorithm. We deduce the convergencecondition and the upper error bound of ObSTP algorithm. The theoreticalresults show that the convergence condition is mainly related to the biorthogonal parameter θ of the matrix $\tilde{\Phi}^*\Phi$ and the RIC $\tilde{\delta}_{2s}$ of the matrix $\tilde{\Phi}$ .Inthe simulation, we illustrate the advantages of theObSTP algorithm in theanisotropic case compared to the STP algorithm. The first experimentalresults show that the ObSTP algorithm outperforms the STP algorithm in terms of recovery performance. In the second experiment, we obtain that the ObSTP algorithm could resist the noise perturbation. The Oblique Pursuitmethod is a new research direction in compressed sensing. Our future workwill continue to focus on the RBOP analysis of new algorithms.

## VI. APPENDIX

If the matrix $\tilde{\Phi}^*\Phi \in \mathbb{R}^{m\times N}$ satisfies the RBOP with parameters $(s,\theta_s)$, by the definition 2, it holds that

$$1-\theta_s \le \lambda_{\min}(\tilde{\Phi}^*\Phi_\Gamma) \le \lambda_{\max}(\tilde{\Phi}^*\Phi_\Gamma) \le 1+\theta_s, \tag{8}$$

for all $\Gamma \subset \{1,2,\cdots,N\}$ such that $|\Gamma| \le s$, where $\lambda_{\min}(\tilde{\Phi}^*\Phi_\Gamma)$ and $\lambda_{\max}(\tilde{\Phi}^*\Phi_\Gamma)$ denote the minimal and maximal eigenvalues of $\tilde{\Phi}^*\Phi_\Gamma$ ,respectively.

Then, the general compressed sensing model that used in the main proof is

$$\text{y=}\Phi\text{x+n} = \Phi\text{x}_\Gamma + \Phi\text{x}_{\overline{\Gamma}} + \text{n=}\Phi\text{x}_\Gamma\text{+n}_1, \tag{9}$$

where $\Phi \in \mathbb{R}^{m\times N}$ denotes the measurement matrix, $\text{n} \in \mathbb{R}^m$ stands for the noise vector and

$\text{n}_1\text{=}\Phi\text{x}_{\overline{\Gamma}}\text{+n}$.

Proof of Theorem 1:

When the ObSTP algorithm proceeds to the $(n-1)$ -th iteration, it is defined in step 7

$$\text{x}^{n-1} = \arg\min_{\text{z}\in\mathbb{R}^N}\{\tilde{\Phi}^*(\text{y-}\Phi^*\text{z}^n) \mid \operatorname{supp}(z) \subseteq \Gamma^{n-1}\},$$

then

$$(\tilde{\Phi}^*(\text{y-}\Phi\text{x}^{n-1}))_{\Gamma^{n-1}} = 0\,. \tag{10}$$

According to the definition in the algorithm 1, $\vartriangle\Gamma$ is the set corresponding to the s largest entries in $\tilde{\Phi}^*(\text{y-}\Phi\text{x}^{n-1})$ ,then

$$\|(\tilde{\Phi}^*(\text{y-}\Phi\text{x}^{n-1}))_\Gamma\|_2 \le \|(\tilde{\Phi}^*(\text{y-}\Phi\text{x}^{n-1}))_{\vartriangle\Gamma}\|_2\,.$$

Removing the common coordinates in sets Γ and △ Γ, we turns out

$$\|(\tilde{\Phi}^*(\text{y-}\Phi\text{x}^{n-1}))_{\Gamma/\vartriangle\Gamma}\|_2 \le \|(\tilde{\Phi}^*(\text{y-}\Phi\text{x}^{n-1}))_{\vartriangle\Gamma/\Gamma}\|_2\,. \tag{11}$$

For $\operatorname{supp}(\text{x}_\Gamma) \subseteq \Gamma$ and $\operatorname{supp}(\text{x}^{n-1}) \subseteq \Gamma^{n-1}$, we have

$$(\text{x}_\Gamma\text{-x}^{n-1})_{\vartriangle\Gamma/(\Gamma\cup\Gamma^{n-1})} = 0. \tag{12}$$

For the right part of inequality (8), we have

$$\|(\tilde{\Phi}^*(\text{y-}\Phi\text{x}^{n-1}))_{\vartriangle\Gamma/\Gamma}\|_2$$

$$
\begin{aligned}
&\overset{(7)}{=} \| (\tilde{\Phi}^*(\mathrm{y}\text{-}\Phi \mathrm{x}^{n-1}))_{\vartriangle\Gamma/(\Gamma\cup\Gamma^{n-1})} \|_2 \\
&= \| (\tilde{\Phi}^*(\Phi \mathrm{x}_\Gamma\text{-}\mathrm{n}_1\text{-}\Phi \mathrm{x}^{n-1}))_{\vartriangle\Gamma/(\Gamma\cup\Gamma^{n-1})} \|_2 \\
&\overset{(9)}{=} \| ((\tilde{\Phi}^*\Phi\text{-I})(\mathrm{x}_\Gamma\text{-}\mathrm{x}^{n-1}) + \tilde{\Phi}^*\mathrm{n}_1)_{\vartriangle\Gamma/(\Gamma\cup\Gamma^{n-1})} \|_2 \\
&\leq \| ((\tilde{\Phi}^*\Phi\text{-I})(\mathrm{x}_\Gamma\text{-}\mathrm{x}^{n-1}) + \tilde{\Phi}^*\mathrm{n}_1)_{\vartriangle\Gamma/\Gamma} \|_2 \\
&\leq \| ((\tilde{\Phi}^*\Phi\text{-I})(\mathrm{x}_\Gamma\text{-}\mathrm{x}^{n-1})_{\vartriangle\Gamma/\Gamma} \|_2 + \| (\tilde{\Phi}^*\mathrm{n}_1)_{\vartriangle\Gamma/\Gamma} \|_2
\end{aligned} \tag{13}
$$

In the step 2, it shows that $\tilde{\Gamma}^n = \Gamma^{n-1} \bigcup \vartriangle\Gamma$. For $\mathrm{supp}(\mathrm{x}^{n-1}) \subseteq \Gamma^{n-1} \subseteq \tilde{\Gamma}^n$, we observe that

$$
(\mathrm{x}_\Gamma\text{-}\mathrm{x}^{n-1})_{\Gamma/\tilde{\Gamma}^n} = (\mathrm{x}_\Gamma)_{\overline{\tilde{\Gamma}^n}}. \tag{14}
$$

For the left part of inequality (8), we show that

$$
\begin{aligned}
&\| (\tilde{\Phi}^*(\mathrm{y}\text{-}\Phi \mathrm{x}^{n-1}))_{\vartriangle\Gamma/\Gamma} \|_2 \\
&\overset{(10)}{=} \| (\tilde{\Phi}^*(\mathrm{y}\text{-}\Phi \mathrm{x}^{n-1}))_{\Gamma/(\vartriangle\Gamma\cup\Gamma^{n-1})} \|_2 \\
&= \| (\tilde{\Phi}^*(\Phi \mathrm{x}_\Gamma\text{-}\mathrm{n}_1\text{-}\Phi \mathrm{x}^{n-1}))_{\Gamma/\tilde{S}_n} \|_2 \\
&\overset{(14)}{=} \| ((\tilde{\Phi}^*\Phi\text{-I})(\mathrm{x}_\Gamma\text{-}\mathrm{x}^{n-1})_{\Gamma/\tilde{\Gamma}^n} + (\mathrm{x}_\Gamma)_{\overline{\tilde{\Gamma}^n}} + (\tilde{\Phi}^*\mathrm{n}_1)_{\Gamma/\tilde{\Gamma}^n} \|_2 \\
&\geq \| (\mathrm{x}_\Gamma)_{\overline{\tilde{\Gamma}^n}} \|_2 - \| (\tilde{\Phi}^*\mathrm{n}_1)_{\Gamma/\tilde{\Gamma}^n} \|_2 - \| ((\tilde{\Phi}^*\Phi\text{-I})(\mathrm{x}_\Gamma\text{-}\mathrm{x}^{n-1}))_{\Gamma/\tilde{\Gamma}^n} \|.
\end{aligned} \tag{15}
$$

Combining (11), (13) and (15), we can get that

$$
\begin{aligned}
&\| (\mathrm{x}_\Gamma)_{\overline{\tilde{\Gamma}^n}} \|_2 \\
&\leq \| (\tilde{\Phi}^*\Phi\text{-I})(\mathrm{x}_\Gamma\text{-}\mathrm{x}^{n-1})_{\vartriangle\Gamma/\Gamma} \|_2 + \| ((\tilde{\Phi}^*\Phi\text{-I})(\mathrm{x}_\Gamma\text{-}\mathrm{x}^{n-1})_{\Gamma/\tilde{\Gamma}^n} \| + \| (\tilde{\Phi}^*\mathrm{n}_1)_{\vartriangle\Gamma/\Gamma} \|_2 \\
&+ \| (\tilde{\Phi}^*\mathrm{n}_1)_{\Gamma/\tilde{\Gamma}^n} \|_2 \\
&\leq \sqrt{2} \| ((\tilde{\Phi}^*\Phi\text{-I})(\mathrm{x}_\Gamma\text{-}\mathrm{x}^{n-1})_{(\vartriangle\Gamma/\Gamma)\cup(\Gamma/\tilde{\Gamma}^n)} \|_2 + \sqrt{2} \| (\tilde{\Phi}^*\mathrm{n}_1)_{(\vartriangle\Gamma/\Gamma)\cup(\Gamma/\tilde{\Gamma}^n)} \|_2 \\
&\leq \sqrt{2} \| ((\tilde{\Phi}^*\Phi\text{-I})(\mathrm{x}_\Gamma\text{-}\mathrm{x}^{n-1})_{\vartriangle\Gamma/\Gamma} \|_2 + \sqrt{2} \| (\tilde{\Phi}^*\mathrm{n}_1)_{\vartriangle\Gamma\cup\Gamma} \|_2 \\
&\leq \sqrt{2}\theta_{3s} \| \mathrm{x}_\Gamma\text{-}\mathrm{x}^{n-1} \|_2 + \sqrt{2(1+\tilde{\delta}_{2s})} \| \mathrm{n}_1 \|_2.
\end{aligned} \tag{16}
$$

The step 6 is an identification process. According to the definition of set $S$ and set $S^n$ in the algorithm, we have

$$
\| (\mathrm{u}^n - \mu\tilde{\Phi}^*(\mathrm{y}\text{-}\Phi \mathrm{u}^n)_\Gamma) \|_2 \leq \| (\mathrm{u}^n - \mu\tilde{\Phi}^*(\mathrm{y}\text{-}\Phi \mathrm{u}^n)_{\Gamma^n}) \|_2 . \tag{17}
$$

Removing the common coordinates in sets $\Gamma$ and $\Gamma^n$,

$$
\| (\mathrm{u}^n - \mu\tilde{\Phi}^*(\mathrm{y}\text{-}\Phi \mathrm{u}^n)_{\Gamma/\Gamma^n}) \|_2 \leq \| (\mathrm{u}^n - \mu\tilde{\Phi}^*(\mathrm{y}\text{-}\Phi \mathrm{u}^n)_{\Gamma^n/\Gamma}) \|_2 . \tag{18}
$$

For the right part of inequality (15),

$$
\begin{aligned}
&\| (\mathrm{u}^n - \mu\tilde{\Phi}^*(\mathrm{y}\text{-}\Phi \mathrm{u}^n)_{\Gamma^n/\Gamma}) \|_2 \\
&= \| (\mathrm{u}^n - \mu\tilde{\Phi}^*(\mathrm{x}_\Gamma\text{-}\mathrm{u}^n) + \mu\tilde{\Phi}^*\mathrm{n}_1)_{\Gamma^n/\Gamma} \|_2 \\
&\leq \| ((\mu\tilde{\Phi}^*\Phi\text{-I})((\mathrm{x}_\Gamma - \mathrm{u}^n))_{\Gamma^n/\Gamma}) \|_2 + \| (\mu\tilde{\Phi}^*\mathrm{n}_1)_{\Gamma^n/\Gamma} \|_2 .
\end{aligned}
$$

For the left part of inequality (15) and $(\mathrm{x}_\Gamma)_{\Gamma/\Gamma^n} = (\mathrm{x}_\Gamma)_{\overline{\Gamma^n}}$, we can derive that

$$
\begin{aligned}
&\| (\mathrm{u}^n - \mu\tilde{\Phi}^*(\mathrm{y}\text{-}\Phi \mathrm{u}^n)_{\Gamma/\Gamma^n}) \|_2 \\
&= \| (\mathrm{u}^n - \mu\tilde{\Phi}^*(\mathrm{x}_\Gamma\text{-}\mathrm{u}^n) + \mu\tilde{\Phi}^*\mathrm{n}_1 - \mathrm{x}_\Gamma + \mathrm{x}_\Gamma)_{\Gamma^n/\Gamma}) \|_2 \\
&\geq \| (\mathrm{x}_\Gamma)_{\overline{\Gamma}^n} \|_2 - \| ((\mu\tilde{\Phi}^*\Phi - \mathrm{I})((\mathrm{x}_S\text{-}\mathrm{u}^n))_{\Gamma/\Gamma^n} \|_2 - \| (\mu\tilde{\Phi}^*\mathrm{n}_1)_{\Gamma/\Gamma^n} \|_2,
\end{aligned} \tag{19}
$$

then there is

$$
\begin{aligned}
&\| (\mathrm{x}_\Gamma)_{\overline{\Gamma^n}} \|_2 \\
&\leq \| ((\mu\tilde{\Phi}^*\Phi\text{-I})((\mathrm{x}_\Gamma - \mathrm{u}^n))_{\Gamma^n/\Gamma}) \|_2 + \| (\mu\tilde{\Phi}^*\mathrm{n}_1)_{\Gamma^n/\Gamma} \|_2 \\
&+ \| (((\mu\tilde{\Phi}^*\Phi\text{-I})(\mathrm{x}_\Gamma - \mathrm{u}^n))_{\Gamma/\Gamma^n}) \|_2 + \| (\mu\tilde{\Phi}^*\mathrm{n}_1)_{\Gamma/\Gamma^n} \|_2 \\
&\leq \sqrt{2} \| ((\mu\tilde{\Phi}^*\Phi\text{-I})(\mathrm{x}_\Gamma\text{-}\mathrm{u}^n))_{(\Gamma^n/\Gamma)\cup(\Gamma/\Gamma^n)} \|_2 + \sqrt{2} \| (\mu\tilde{\Phi}^*\mathrm{n}_1)_{(\Gamma^n/\Gamma)\cup(\Gamma/\Gamma^n)} \|_2 \\
&\leq \sqrt{2}(|\mu - 1| - \mu\theta) \| \mathrm{x}_\Gamma\text{-}\mathrm{u}^n \|_2 + \sqrt{2(1+\tilde{\delta}_{2s})}\mu \| \mathrm{n}_1 \|_2.
\end{aligned} \tag{20}
$$

In step 3, it shows that

$$
\tilde{\mathrm{x}}^n = \arg\min_{z\in\mathbb{R}^N} \{ \| \tilde{\Phi}^*(\mathrm{y}\text{-}\Phi \mathrm{z}) \|_2, \mathrm{z} \in \tilde{\Gamma}^n \}
$$

Therefore, by the definition 3, it follows that

$(\tilde{\Phi}^*_{\tilde{\Gamma}^n}\Phi)^{-1}\tilde{\Phi}^*_{\tilde{\Gamma}^n}(\text{y-}\Phi\tilde{\text{x}}^n)=0$,

By the RBOP of the matrix $\tilde{\Phi}^*\Phi$, $\tilde{\Phi}^*\Phi$ has full rank. Hence,

$\tilde{\Phi}^*_{\tilde{\Gamma}^n}((\text{y-}\Phi\tilde{\text{x}}^n)=\tilde{\Phi}^*_{\tilde{\Gamma}^n}(\Phi(\text{x}_\Gamma-\tilde{\text{x}}^n))+\text{n}_1)=0.$

then it turns that

$\tilde{\Phi}^*\Phi(\text{x-x}_{\tilde{\Gamma}^n})_{\tilde{\Gamma}^n}=\tilde{\Phi}^*\text{n}_1.$

Now, it has that

$$\|(\text{x}_\Gamma-\tilde{\text{x}}^n)_{\tilde{\Gamma}^n}\|_2^2$$
$$=\|(\text{x}_\Gamma-\tilde{\text{x}}^n)_{\overline{\tilde{S}^n}}\|_2^2+\|(\text{x}_\Gamma-\tilde{\text{x}}^n)_{\tilde{\Gamma}^n}\|_2^2$$
$$=\|(\text{x}_\Gamma)_{\overline{\tilde{\Gamma}^n}}\|_2^2+\|(\text{x}_\Gamma-\tilde{\text{x}}^n)_{\tilde{\Gamma}^n}\|_2^2$$
$$\leq\|(\text{x}_\Gamma)_{\overline{\tilde{\Gamma}^n}}\|_2^2+(\theta\|(\text{x}_\Gamma-\tilde{\text{x}}^n)\|_2+\sqrt{1+\tilde{\delta}_{2S}}\|\text{n}_1\|_2)^2.$$

By solving the quadratic equation, it has that

$$\|(\text{x}_\Gamma-\tilde{\text{x}}^n)\|_2^2\leq\sqrt{\frac{1}{1-\theta^2}}\|(\text{x}_\Gamma)_{\overline{\tilde{\Gamma}^n}}\|_2+\frac{\sqrt{1+\tilde{\delta}_{2S}}}{1+\theta}\|\text{n}_1\|_2,$$

then

$$\|\text{x}_\Gamma\text{-}\tilde{\text{x}}^n\|_2$$
$$\leq\sqrt{\frac{2\theta^2}{1-\theta}}\|\text{x}_\Gamma-\text{x}^{n-1}\|_2+(\sqrt{\frac{1}{1-\theta^2}}\sqrt{2(1+\tilde{\delta}_{2S})}+\frac{\sqrt{1+\tilde{\delta}_{2S}}}{1-\theta}\|\text{n}_1\|_2).$$

Define $\Gamma_\nabla=\tilde{\Gamma}^n/U^n$ ,then

$$\|(\text{x}_\Gamma)_{\Gamma_\nabla}\|_2\leq\sqrt{2}\theta\|(\text{x}_\Gamma-\tilde{\text{x}}^n)\|_2+\sqrt{2(1+\tilde{\delta}_{2S})}\|\text{n}_1\|_2).$$

Dividing $\overline{U^n}$ into two disjoint parts $\Gamma_\nabla$ and $\overline{\tilde{\Gamma}^n}$ and assuming $t_1=\sqrt{\frac{1}{1-\theta^2}}\sqrt{2(1+\tilde{\delta}_{2S})}+\frac{\sqrt{1+\tilde{\delta}_{2S}}}{1-\theta}$,

$t_2=\sqrt{1+\tilde{\delta}_{2S}}$, it has

$$\|(\text{x}_\Gamma)_{\overline{U^n}}\|_2^2$$
$$=\|(\text{x}_\Gamma)_{\Gamma_\nabla}\|_2^2+\|(\text{x}_\Gamma)_{\overline{\tilde{\Gamma}^n}}\|_2^2$$
$$\leq 2(\theta\|\text{x}_\Gamma-\tilde{\text{x}}^n\|_2+t_1\|\text{n}_1\|_2)^2+2(\delta_{3s}\|\text{x}_\Gamma-\tilde{\text{x}}^n\|_2+t_2\|\text{n}_1\|_2)^2$$
$$\leq 2(\sqrt{\frac{2\theta^2}{1-\theta^2}}\|\text{x}_\Gamma-\text{x}^{n-1}\|_2+(\theta t_1+t_2)\|\text{n}_1\|_2)^2+2(\theta\|\text{x}_\Gamma-\text{x}^{n-1}\|_2+t_2\|\text{n}_1\|_2)^2$$
$$\leq 2(\sqrt{\frac{2\theta^4}{1-\theta^2}+\theta^2}\|\text{x}_\Gamma-\text{x}^{n-1}\|_2+((\theta t_1+t_2)+t_2)\|\text{n}_1\|_2)^2$$
$$\leq 2(\sqrt{\frac{\theta^2(1+\theta^2)}{1-\theta^2}+\theta^2}\|\text{x}_\Gamma-\text{x}^{n-1}\|_2+(\theta t_1+2t_2)\|\text{n}_1\|_2)^2$$

then

$$\|(\text{x}_\Gamma)_{\overline{U^n}}\|_2\leq\sqrt{\frac{2\theta^2(1+\theta^2)}{1-\theta^2}}\|\text{x}_\Gamma-\text{x}^{n-1}\|_2+\sqrt{2}(\theta t_1+2t_2)|\text{n}_1\|_2,$$

and

$$\|(\text{x}_\Gamma-\text{u}^n)_{\overline{U^n}}\|_2\leq\|(\text{x}_\Gamma-\tilde{\text{x}}^n)_{\tilde{\Gamma}^n}\|_2$$
$$\leq\theta\|\text{x}_S-\tilde{\text{x}}^n\|_2+\sqrt{1+\tilde{\delta}_{2S}}|\text{n}_1\|_2$$
$$\leq\sqrt{\frac{2\theta^4}{1-\theta^2}}\|\text{x}_\Gamma-\text{x}^{n-1}\|_2+(\theta t_1+2t_2)\|\text{n}_1\|_2).$$

For $\text{supp}(\text{x}_\Gamma-\text{u}^n)=U^n\bigcup\overline{U^n}$, $\text{supp}(\text{u}^n)\subseteq U^n$ and $(\text{x}_\Gamma-(\text{u}^n)_{\overline{U^n}}=(\text{x}_\Gamma)_{\overline{U^n}}$,

we can get that

$$\|\text{x}_\Gamma-\text{u}^n\|_2^2=\|(\text{x}_\Gamma-\text{u}^n)_{U^n}\|_2^2+\|(\text{x}_\Gamma-\text{u}^n)_{\overline{U^n}}\|_2^2$$
$$=\|(\text{x}_\Gamma-\text{u}^n)_{U^n}\|_2^2+\|(\text{x}_\Gamma)_{\overline{U^n}}\|_2^2$$
$$\leq\sqrt{\frac{2\theta^4}{1-\theta^2}}\|\text{x}_\Gamma-\text{x}^{n-1}\|_2+(\theta t_1+t_2)\|\text{n}_1\|_2)^2+(\sqrt{\frac{2\theta^2(1+\theta^2)}{1-\theta^2}}\|\text{x}_\Gamma-\text{x}^{n-1}\|_2+\sqrt{2}(\theta+2t_2)|\text{n}_1\|_2)^2$$
$$\leq(\sqrt{\frac{2\theta^2(1+2\theta^2)}{1-\theta^2}}\|\text{x}_\Gamma-\text{x}^{n-1}\|_2+(\sqrt{2}+1)\theta t_1+(2\sqrt{2}+1)t_2)\|\text{n}_1\|_2)^2$$
$$\|\text{x}_\Gamma-\text{u}^n\|_2$$

$$\leq (\sqrt{\frac{2\theta^2(1+2\theta^2)}{1-\theta^2}}\|x_\Gamma - x^{n-1}\|_2 + (\sqrt{2}+1)\theta t_1 + (2\sqrt{2}+1)t_2) \| n_1 \|_2)^2,$$

$$\| (x_\Gamma)_{\overline{\Gamma^n}} \|_2 \leq \sqrt{2}(|\mu-1|+\mu\theta) \| x_\Gamma - u^n \|_2 + \sqrt{2(1+\tilde{\delta}_{2s})}\mu \| n_1 \|_2 ,$$

and

$$| x_\Gamma - x^n \|_2 \leq \sqrt{\frac{1}{1-\theta^2}} \| (x_\Gamma)_{\overline{\Gamma^n}} \|_2 + \frac{\sqrt{1+\tilde{\delta}_{2s}}}{1-\theta} \| n_1 \|_2$$

Combining (20), (21), and (22), it turns out that

$$\begin{aligned}
&\| x_\Gamma - u^n \|_2 \\
&\leq \sqrt{\frac{1}{1-\theta^2}}(\sqrt{2}\,|\mu-1|+\mu\theta) \| x_\Gamma - u^n \|_2 + \sqrt{2(1+\tilde{\delta}_{2s})}\mu \| n_1 \|_2) + \frac{\sqrt{1+\tilde{\delta}_{2s}}}{1-\theta} \| n_1 \|_2 \\
&\leq \sqrt{\frac{1}{1-\theta^2}}\{\sqrt{2}(|\mu-1|+\mu\theta)[\sqrt{\frac{2\theta^2(1+2\theta^2)}{1-\theta^2}}\|x_\Gamma - x^{n-1}\|_2 \| n_1 \|_2 \\
&+(\sqrt{2}+1)\theta t_1 + (2\sqrt{2}+1)t_2] + \sqrt{2(1+\theta)}\mu \| n_1 \|_2 + \frac{\sqrt{1+\tilde{\delta}_{2s}}}{1-\theta} \| n_1 \|_2 \\
&= \rho \| x_\Gamma - x^{n-1} \|_2 + (1-\rho)C \| n_1 \|_2,
\end{aligned}$$

where $\rho = \frac{2\theta(|\mu-1|+\mu\theta\sqrt{1+2\theta^2}}{1-\theta^2}$ and

$$(1-\rho)C = \sqrt{\frac{1}{1-\theta^2}}[((\sqrt{2}+1)\theta t_1 + (2\sqrt{2}+1)t_2 + \sqrt{2(1+\theta)}\mu + \frac{\sqrt{1+\tilde{\delta}_{2s}}}{1-\theta}].$$

By recursive method, we have

$$\| x_\Gamma - x^n \|_2 \leq \rho^n \| x_\Gamma \|_2 + C \| n_1 \|_2 .$$

In order to ensure that the sequences $x^n$ which generated from the ObSTPalgorithm are convergent, it only need $\rho < 1$. However, the parameter $\rho$ is associated with $\theta$ and $\mu$ .The following three cases are the equivalentconditions $\rho < 1$. 1) When $\mu < 1, \rho < 1$ is equivalent to $\frac{1+\theta}{2\theta\sqrt{1+2\theta}} < \mu < 1$; 2) When $\mu > 1, \rho < 1$ is equivalent to $0 < \mu < \frac{1}{1+\theta} + \frac{1-\theta}{2\theta\sqrt{1+2\theta^2}}$; 3) When $\mu = 1, \rho < 1$ is equivalent to $\rho = \frac{2\theta^2\sqrt{1+\theta^2}}{1-\theta^2} < 1$, thus finally we can get that $\theta < 0.535$.

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