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Group Consensus of Heterogenous Continuous-time Multi-agent Systems

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Abstract: This paper considers the group consensus problem for continuous-time linear heterogeneous multi-agent systems with undirected and directed fixed topology. In order to obtain group consensus, we use two partition coefficients to divide all second-order agents and all first-order agents as the two groups, a novel protocol is designed. By constructing the Lyapunov function, a sufficient condition for group consensus under undirected topology are proved. Based on a system transformation method, the group consensus for heterogeneous multi-agent systems is transformed into a group consensus for homogeneous multi-agent systems. We also find the convergence points of the two groups, it has great significance. Finally, numerical examples are provided to demonstrate the effectiveness of the theoretical results.

Keywords: Group consensus; heterogeneous multi-agent systems; undirected topology; directed topology

Introduction

In recent years, distributed multi-agent cooperative control system has been widely applied in the fields of unmanned spacecraft cooperative control, control of formation flying satellites, mobile robot distributed optimization, and as a basic problem of multi-agent system cooperative control, consensus problem of multi-agent systems got many researchers attention[1]-[5]. Using graph theory, Jadbabaie et al.[6] provided a theoretical explanation for the consensus behavior of the Vicsek model. Bases on the analysis in[6], Saber et al. [7] investigated the consensus problems for networks of dynamic agents with fixed and switching topologies by discussing three cases: directed networks with fixed topology, directed networks with switching topology, undirected networks with communication time-delays and fixed topology. Ref.[8] established a necessary and sufficient second-order consensus criterion and proved that both the real and imaginary parts of the eigenvalues of the Laplacian matrix play key roles in roles in reaching consensus. Consensus problem is the design of a suitable control protocol so that all the multi-agent achieve to the same value through a certain amount.

In real life, group collaboration is a very common phenomenon, and it is very important. It has many civil and military applications in surveillance, reconnaissance, battle field assessment, etc. Therefore, re-search on the group consensus not only helps better understand the mechanisms of natural collective phenomena, but also provides some

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useful ideas for distributed cooperative control. In the group consensus problem, the whole network is divided into multiple sub-networks with information exchanges between them, and the aim is to design appropriate protocol such that agents in the same sub-networks reach the same consistent states. It is easy to see that consensus problem in [9]-[11] is a special case of group consensus. It described the system consensus in one group. However, there are more groups in the practical example [12]-[13]. In [12], Yu and Wang studied the group consensus in multi-agent systems with switching topologies and communication delays, where the agents are described by single-integrator dynamics, they introduce double-tree-form transformations under which dynamic equations of agents are transformed into reduced-order systems. Group consensus control for second-order dynamic multi-agent systems was investigated in[14]-[16], [16] studied the group consensus problem of second-order multi-agent systems with time delays, where found the upper bound of time delay such that the multi-agent systems can achieve group consensus.

From the above, most of existing research results about consensus problem of multi-agent systems are established mainly based on the homogeneous multi-agent systems, that is, all the agents have the same dynamics behaviours. However, the dynamics of each coupling agents may be different in practice work, because of the external impact or restrictions of exchanging. Therefore, it has very important practical significance that study group consensus problem for heterogeneous multi-agent system. Some existing literatures about consensus problems of heterogeneous multi-agent systems mainly consider the mix of first-order and second-order systems.[17]-[18] considered consensus control for a class of heterogeneous multi-agent systems, some conditions were presented for heterogeneous multi-agent systems with fixed and switching directed communication topology to reach consensus by using tools from graph theory and matrix theory. In[19], Zheng and Wang presented a novel protocol according to the feature of heterogeneous multi-agent, studied the group consensus in continuous-time system.[20] studied consensus problem of heterogeneous multi-agent in continuous-time system and discrete-time system by a novel consensus algorithm, respectively.

Inspired by the aforementioned work, in this paper, we will investigate the group consensus of heterogeneous multi-agent systems under undirected and directed fixed topology in continuous-time. The main contributions of this paper are two folds: (1) Two partition coefficients are introduced to realize the group consensus of heterogeneous multi-agent system; (2) The concrete convergence points are given.

This paper is organized as follows. Section 2 introduces some basic concepts and notations in graph theory and presents the models of the problems. Section 3 gives the main results of heterogeneous multi-agent systems under undirected and directed fixed topology in continuous-time. Some numerical simulations of the theoretical results are given in Section 4. Finally, the conclusion is made in section 5.

1. Preliminaries

1.1 Graph Theory

Graph theory[21] is an effective tool to study the coupling topology of the communication configuration of the agents. In this section, we briefly review some basic notations and concepts in graph theory that will be used in this paper [22].

A weighted directed graph G=(V, E, A) consists of a vertex set V= $\{v_1, v_2,\cdots, v_n\}$ and an edge set E= $\{(v_i, v_j):v_i, v_j\in V\}$, where an edge is an ordered pair of distinct vertices of V, and the non symmetric weighted adjacency matrix A= $[a_{ij}]$, with $a_{ij}>0$, if and only if $e_{ij}\in E$ and $a_{ij}=0$ if not. If all the elements of V are unordered pairs, then the graph is called an undirected graph. If $v_i, v_j\in V$, and $(v_i, v_j)\in E$, then we say that v_i and v_j are adjacent or v_j is a neighbor of v_i. The neighborhood set of node v_i is denoted by $N_J=\{v_j\in V:(v_i, v_j)\in E\}$. The number of neighbors of each vertex is its degree.

A graph is called complete if every pair of vertices are adjacent. A path of length r from v_i to v_j in a graph is a sequence of r+1 distinct vertices starting with viand ending with v_j such that consecutive vertices are adjacent. If there is a path between any two vertices of G, then G is connected. The degree matrix D(G) of G is a diagonal matrix with rows and columns indexed by V, in which the (v_i, v_j) entry is the degree of vertex v_i. The symmetric matrix defined as.

$$L(G) = D(G) - A(G)$$

is the Laplacian of G. The Laplacian is always symmetric and positive semi-definite, and the algebraic multiplicity of its zero eigenvalue is equal to the number of connected components in the graph.

Definition 1 [12] (Subgraph)

A network with topology G_1=(V_1, E_1, A_1) is said to be a sub-network of a network with topology G=(V, E, A) if (i)$V_1 \subseteq V$, (ii)$E_1 \subseteq E$ and (iii) the weighted adjacency matrix A_1 inherits A . Correspondingly, we call G_1 a subgraph of G. Furthermore, if the inclusion relations in (i) and (ii) are strict, and E1={ (v_i, v_j) : i， j$\in V_1$, (v_i, v_j) : i， j$\in$E}, we say that the first network is a proper sub-network of the second one. Correspondingly, we call G_1 is a proper subgraph of G.

In reality, the whole multi-agent system can be divided into some complex subgroups or intelligent clusters, which can make the cooperation and control of such systems have a more practical significance. Without loss of generality, all the agents are in the network (G, x) consisting of n(n＞1) agents divided into two different subgroups (G_1, x^1) consisting of m second-order (double) integrator agents and (G_2, x^2) consisting of n-m first-order (single) integrator agents with $x^1=(x_1, x_2,\ldots x_m)^T$ and $x^2=(x_{m+1}, x_{m+2},\ldots, x_n)^T$, and the agents in each subgroup can establish a sub network. An example of two different subgroups connected digraph is displayed in Fig. 1.

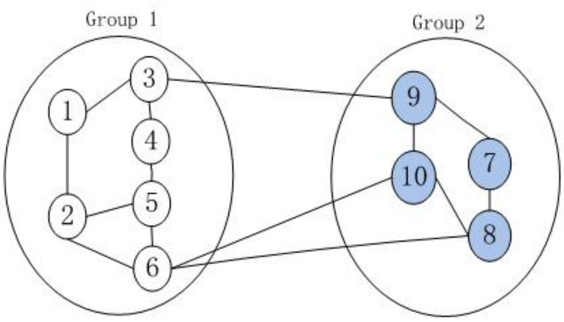

Fig. 1: A undirected topology.

The group consensus problem is studied under first-order and second-order integrator agents obeying the neighbor-based law. The position adjacency matrix of the network can be partitioned as

$$A = \begin{pmatrix} A_s & A_{sf} \\ A_{fs} & A_f \end{pmatrix}$$

Where A_s, A_f, A_{sf}, and A_{fs} correspond to, respectively, the indices of second-order integrator agents, first-order integrator agents, from second-order integrator agents to first-order integrator agents and from first-order integrator agents to second-order integrator agents. In what follows, with respect to corresponding Laplacian matrices and degree matrices, we denote by L_s, L_f and D_s, D_f, D_{sf} and D_{fs}, respectively. Similarly, the Laplacian matrix of the network can be partitioned as

$$L = \begin{pmatrix} \overline{L}_s & -A_{sf} \\ -A_{fs} & \overline{L}_f \end{pmatrix}$$

where $\overline{L}_s \quad L_s + D_{sf}$, $\overline{L}_f \quad L_f + D_{fs}$. The Laplacian matrix of corresponding to velocity adjacency matrix of the network is denoted by $\hat{L}$

1.2 System model

In this subsection, we propose the heterogeneous multi-agent system which is composed of a CT system

$$\begin{cases} x_i(t) = v_i(t), v_i(t) = u_i(t), & i \in \mathrm{I}_m, \\ x_i(t) = u_i(t), & i \in \mathrm{I}_n / \mathrm{I}_m, \end{cases} \quad .$$

Table (1-1)

where $x_i \in$ R, $v_i \in$ R and $u_i \in$ R and are the position-like, velocity-like and control input, respectively, of agent *i*. The interactions among agents are realized through the following protocol,

$$u_i(t) = \begin{cases} \sum_{j=1}^{n} a_{ij}(h_l x_j - h_2 x_i) + k_1 h_2 \sum_{j=1}^{m} b_{ij}(v_j - v_i), i \in \mathrm{I}_m, l \in \{1,2\}, \\ k_2 \sum_{j=1}^{n} a_{ij}(h_l x_j - h_1 x_i), i \in \mathrm{I}_n / \mathrm{I}_m, l \in \{1,2\}. \end{cases}$$

Table(1-2)

where $k_1 > 0$, $k_2 > 0$ are the feedback gains, $h_1 > 0$, $h_2 > 0$, A=［a_{ij}］, is about position communication of agent *i* and *j*, B=［b_{ij}］ is about velocity communication of agent *i* and *j*.

Remark 1 h_1, h_2 are constants being used to partition the sub-groups consisting of first-order integrator agents and second-order integrator agents, respectively.

Definition 2 The heterogeneous multi-agent continuous-time system (1) is said to reach group consensus if for any initial conditions x_0 and v_0, it follows that

$$\begin{cases} \lim_{t \to \infty} \| x_i(t) - x_j(t) \| = 0, i, j \in \mathrm{I}_n, \\ \lim_{t \to \infty} \| v_i(t) - v_j(t) \| = 0, i, j \in \mathrm{I}_m. \end{cases}$$

Remark 2 In the following analysis, for notational simplicity, we only consider the case, i.e., all agents are assumed in one-dimensional space. However, all results we have obtained can be easily extended for n-dimensional space by using the Kronecker product.

2. Group consensus of heterogeneous multi-agent system

In this section, we consider the group consensus of the heterogeneous multi-agent systems composed of a CT system (1) under undirected and directed fixed topology. Some sufficient and/or necessary conditions will be obtained for solving the group consensus problem. For the convenience of discussion, let

$$\mathbf{x}_s \quad [x_1, x_2, \quad, x_m], \mathbf{v}_s \quad [v_1, v_2, \quad, v_m], \ \mathbf{x}_f \quad [x_{m+1}, x_{m+2}, \quad, x_n].$$

The initial condition are $x_i(0)=x_{i0}$, $v_i(0)=v_{i0}$. Let $x_0=[x_{10}, \ x_{20}, \cdots x_{n0}]^T$, $v_0=[v_{10}, \ v_{20}, \cdots v_{n0}]^T$.

Let $p_i=h_2x_i$, $q_i=h_2v_i$, $i\in I_m$, and $p_i=h_1x_i$, $i\in I_N/I_M$. Thus, the heterogeneous multi-agent system (1) with protocol (2) can be rewritten as follows,

$$\begin{cases} p_i(t)=q_i(t), i\in I_m, \\ q_i(t)=h_2\sum_{j=1}^{n}a_{ij}(p_j-p_i)+k_1h_2\sum_{j=1}^{m}b_{ij}(q_j-q_i), i\in I_m, \\ p_i(t)=k_2h_1\sum_{j=1}^{n}a_{ij}(p_j-p_i), i\in I_n/I_m. \end{cases}$$

Table（1-3）

Obviously, system (2-1) with protocol (2-2) can solve the group consensus if and only if the system (2-3) can achieve the group consensus.

For the undirected graph, under the symmetry condition of the weight matrices A and B, that is, $A=A^T$ and $B=B^T$, one can obtain the following result.

Theorem 1 For the connected undirected G with $A=A^T$ and $B=B^T$, system (1) steered by (2) can solve the group consensus problem.

Proof. Take the Lyapunov function,

$$V(t)=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\frac{(p_j(t)-p_i(t))^2}{2}+\sum_{i=1}^{m}\frac{(q_i(t))^2}{h_2},$$

which is positive definite with respect to $p_j(t)-p_i(t)(\forall i\neq j, i, j\in I_n)$ and $q_i(t)(i\in I_m)$.

Differentiating $V(t)$, yields

$$\begin{aligned} V(t)&=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}(p_j-p_i)(p_j-p_i)+\sum_{i=1}^{m}\frac{2q_iq_i}{h_2} \\ &=\sum_{i=1}^{m}2q_i\left(\sum_{j=1}^{n}a_{ij}(p_j-p_i)+k_1\sum_{j=1}^{m}b_{ij}(q_j-q_i)\right) \\ &+\sum_{i=1}^{m}\sum_{j=1}^{m}a_{ij}(p_j-p_i)(q_j-q_i)+\sum_{i=m+1}^{n}\sum_{j=1}^{m}a_{ij}(p_j-p_i)(q_j-p_i) \\ &+\sum_{i=1}^{m}\sum_{j=m+1}^{n}a_{ij}(p_j-p_i)(p_j-q_i)+\sum_{i=m+1}^{n}\sum_{j=m+1}^{n}a_{ij}(p_j-p_i)(p_j-p_i). \end{aligned}$$

Since $A=A^T$ and $B=B^T$, then

$$\sum_{i=1}^{m}\sum_{j=1}^{m}a_{ij}(p_j-p_i)(q_j-q_i)=-2\sum_{i=1}^{m}\sum_{j=1}^{m}a_{ij}(p_j-p_i)q_i,$$

$$\sum_{i=m+1}^{n}\sum_{j=1}^{m}a_{ij}(p_j-p_i)(q_j-p_i)=\sum_{i=1}^{m}\sum_{j=m+1}^{n}a_{ij}(p_j-p_i)(p_j-q_i),$$

$$\sum_{i=m+1}^{n}\sum_{j=m+1}^{n}a_{ij}(p_j-p_i)(p_j-p_i)=2\sum_{i=m+1}^{n}\sum_{j=m+1}^{n}a_{ij}(p_j-p_i)p_j,$$

$$\sum_{i=1}^{m}\sum_{j=1}^{m}b_{ij}(q_j-q_i)^2=-\sum_{i=1}^{m}\sum_{j=1}^{m}b_{ij}(q_j-q_i)q_i.$$

Hence,

$$\begin{aligned}V(t)&=-2k_1\sum_{i=1}^{m}b_{ij}(q_j-q_i)^2+2\sum_{i=1}^{m}\sum_{j=m+1}^{n}a_{ij}(p_j-p_i)p_j+2\sum_{i=m+1}^{n}\sum_{j=m+1}^{n}a_{ij}(p_j-p_i)p_j\\&=-2k_1\sum_{i=1}^{m}b_{ij}(q_j-q_i)^2+2\sum_{i=1}^{n}\sum_{j=m+1}^{n}a_{ij}(p_j-p_i)p_j\\&=-2k_1\sum_{i=1}^{m}b_{ij}(q_j-q_i)^2-2\sum_{i=m+1}^{n}p_i\sum_{j=1}^{n}a_{ij}(p_j-p_i)\\&=-2k_1\sum_{i=1}^{m}b_{ij}(q_j-q_i)^2-\frac{2}{k_2h_1}\sum_{i=m+1}^{n}p_i^2\ \le 0,\end{aligned}$$

if $k_1>0$, $k_2>0$, $h_1>0$. It follows from Lasalle's invariance principle [23] that

$$\begin{cases}\lim_{t\to\infty}\|p_i(t)-p_j(t)\|=0, i,j\in \mathrm{I}_n,\\ \lim_{t\to\infty}\|q_i(t)-q_j(t)\|=0, i,j\in \mathrm{I}_m.\end{cases}$$

Due to $p_i=h_2x_i$, $q_i=h_2v_i$, $i\in \mathrm{I}_m$, and $p_i=h_1x_i$, $i\in \mathrm{I}_n/\mathrm{I}_m$, we get

$$\begin{cases}\lim_{t\to\infty}h_2\|x_i(t)-x_j(t)\|=0, i,j\in \mathrm{I}_m\\ \lim_{t\to\infty}h_1\|x_i(t)-x_j(t)\|=0, i,j\in \mathrm{I}_n/\mathrm{I}_m,\\ \lim_{t\to\infty}h_2\|v_i(t)-v_j(t)\|=0, i\in \mathrm{I}_m.\end{cases}$$

This completes the proof. In the following, we will consider the the group consensus of system (1) with protocol (2) under directed graph. A necessary lemma [24] is needed.

Lemma 1[24] Suppose that $x=[x_1,x_2,\ \ ,x_n]^T\in \mathrm{R}^n$ and $L=[l_{ij}]\in \mathrm{R}^{n\times n}$ satisfy that $l_{ij}\le 0, i\ne j$ and $\sum_{j=1}^{N}l_{ij}=0, i=1,2,\ \ ,N$. Then, the following conditions are equivalent,

(1) Consensus is reached asymptotically for the system $x=-Lx$;

(2) The directed graph of L has a directed spanning tree, e^{Lt} is a row stochastic matrix with positive diagonal entries for $\forall t\geq 0$, there exists a nonnegative column vector $\mathbf{c}\in R^{n}$, such that $e^{-Lt}\rightarrow\mathbf{1}$ as $t\rightarrow\infty$ where $c^{T}L=\mathbf{0}^{T}$ and $c^{T}\mathbf{1}=1$;

(3) The rank of L is *n-1*;

(4) L has a simple zero eigenvalue and all other eigenvalues have positive real parts;

(5) $Lx=0$ implies that $x_1=x_2=\ldots=x_n$.

From Lemma 1, we have the following result for the directed graph.

Theorem 2 System (1) with protocol (2) can achieve group consensus asymptotically if and only if the graph G contains a directed spanning tree, if the feedbacks gains satisfy $k_1>(1+\frac{1}{h_2})\bar{d}_s, k_2>0$, where $\bar{d}_s=\max_{i=1,\ \ ,m}\sum_{j=1}^{n}a_{ij}$ is maximum value of the diagonal element of corresponding Laplician matrix $\bar{L}_s$.

Proof. Sufficiency:

Let $\mathbf{p}_s\ \ [p_1,p_2,\ \ ,p_m]^T,\mathbf{q}_s\ \ [q_1,q_2,\ \ ,q_m]^T,\mathbf{p}_f\ \ [p_{m+1},p_{m+2},\ \ ,p_n]^T,\xi\ \ [\mathbf{p}_s,\mathbf{q}_s,\mathbf{p}_f]$, then system (3) can be rewritten as

$$\xi(t)=-\begin{pmatrix} O & -I_m & O \\ h_2\bar{L}_s & k_1h_2\hat{L} & -h_2A_{sf} \\ -k_2h_1A_{fs} & O & k_2h_1\bar{L}_f \end{pmatrix}\xi(t)\quad -\Phi\xi(t)$$

Table(1-4)

where $\Phi\ \begin{pmatrix} O & -I_m & O \\ h_2\bar{L}_s & k_1h_2\hat{L} & -h_2A_{sf} \\ -k_2h_1A_{fs} & O & k_2h_1\bar{L}_f \end{pmatrix}$, I is an identity matrix and O is the zero matrix.

Transform matrix Φ using the following nonsingular matrix

$$Q=\begin{pmatrix} I_m & 0 & 0 \\ I_m & I_m & 0 \\ 0 & 0 & I_{n-m} \end{pmatrix},$$

Then we can get

$$\Phi^*=Q\Phi Q^{-1}=\begin{pmatrix} I_m & -I_m & 0 \\ h_2\bar{L}_s-k_1h_2\hat{L}I_m+I_m & k_1h_2\hat{L}I_m-I_m & -h_2A_{sf} \\ -k_2h_1I_{n-m}kA_{fs} & 0 & k_2h_1I_{n-m}\bar{L}_f \end{pmatrix}.$$

Table（1-5）

Let $\eta(t) = Q\xi(t)$, then we can rewrite system (6) as follows:

$\eta(t) = -\Phi^{*}\eta(t)$.

Obviously, Φ^{*} and Φ have the same eigenvalues. Since $k_1 > (1+\frac{1}{h_2})\bar{d}_s, k_2 > 0$, it follows that all diagonal elements of matrix Φ^{*} are nonnegative and all nondiagonal elements are nonpositive. Hence, the matrix Φ^{*} can be seen as a valid Laplacian matrix of graph G^{*}.

Using elementary transformation for Φ^{*}, then

$$\Phi^{*} \rightarrow \begin{pmatrix} -I_m & 0 & 0 \\ 0 & h_2\bar{L}_s & -h_2A_{sf} \\ 0 & -k_2h_1A_{fs} & k_2h_1\bar{L}_f \end{pmatrix} \rightarrow \begin{pmatrix} I_m & 0 & 0 \\ 0 & \bar{L}_s & -A_{sf} \\ 0 & -A_{fs} & \bar{L}_f \end{pmatrix} = \begin{pmatrix} I_m & O \\ O & L \end{pmatrix}.$$

Table（1-6）

Therefore,
$rank(\Phi^{*}) = m + rank(L)$.

From Lemma 1, we have know that $rank(L) = n-1$. Since G contains a directed spanning tree, and from (2-6), we can obtain that $rank(\Phi^{*}) = m+n-1$. Hence, graph G^{*} also has a directed spanning tree. Therefore, system (2-5) reaches consensus asymptotically, and then the heterogeneous multi-agent system (2-3) can achieve the group consensus.

Necessity: By contradiction, assuming that G has no a directed spanning tree, then G also has no a directed tree. From Lemma 1, we can easily know system (5) can not reaches consensus asymptotically, which contradicts to the condition of Theorem 2. This completes the proof of necessity.

In general, most of the results of existing literatures only show that the convergence of the system (1), but don't pay much attention to the specific convergence points. In the following, we will consider this problem.

Theorem 3 If system (1) with protocol (2) can achieve group consensus asymptotically, then the conver-gence points are

$$\lim_{t\to\infty} x_s = \frac{1}{h_2}(k_1k_2\mathbf{1}_m h_1h_2\mu_1^T\hat{L}x_s(0) + k_2\mathbf{1}_m h_1\mu_1^T v_s(0) + \mathbf{1}_m h_2\mu_2^T x_f(0)) + 1,$$

$$\lim_{t\to\infty} v_s = \mathbf{1}_m 0,$$

$$\lim_{t\to\infty} x_f = \frac{1}{h_1}(k_1k_2\mathbf{1}_{n-m} h_1h_2\mu_1^T\hat{L}x_s(0) + k_2\mathbf{1}_{n-m} h_1\mu_1^T v_s(0) + \mathbf{1}_{n-m} h_2\mu_2^T x_f(0)) + 1\cdot$$

Proof. First of all, based on the meaning of h_1, h_2 of the group consensus in this paper, we have that system (1) with protocol (2) can achieve group consensus, as $t\to\infty$ if

$$\begin{matrix} x_i - h_i \to c, l \in \{1,2\}, i \in \mathrm{I}_n \\ v_i \to 0, i \in \mathrm{I}_m \end{matrix},$$

Table（1-7）

where c is a constant. It means that the nodes in the two groups can reach a consensus state asymptotically while there is no consensus among different groups.

According to Lemma 1, we can know that matrix Φ has only one eigenvalue 0, and the real part of non-zero eigenvalues are all greater than or equal to 0. Let J is the Jordan canonical form of matrix $-\Phi$, then we have

$$-\Phi = PJP^{-1} = \left(\omega_1, \quad, \omega_{n+m}\right)\begin{pmatrix} 0 & 0_{1\times(n+m-1)} \\ 0_{(n+m-1)} & J' \end{pmatrix}\begin{pmatrix} \gamma_1^T \\ \\ \gamma_{n+m}^T \end{pmatrix}.$$

Without loss of generality, we choose $\omega_1 = \left[1_m^T, 0_m^T, 1_{n-m}^T\right]^T$, and easily know ω_1 is a right eigenvector of eigenvalue 0 of matrix Φ. The directed topology G contains a spanning tree, so matrix L only has one 0 eigenvalue. And we know there exists an non-negative vector $\mu^T = [\mu_1^T, \mu_2^T] \in \mathrm{R}^n$ with $\mu^T L = 0$, then

$$\begin{cases} \mu_1^T \overline{L}_s - \mu_2^T A_{fs} = 0, \\ -\mu_1^T A_{sf} + \mu_2^T \overline{L}_f = 0. \end{cases}$$

According to $\gamma_1 \Phi = \gamma_1 \lambda$, we can get

$$\gamma_1^T = \left[k_1 k_2 h_1 h_2 \mu_1^T \hat{L}, k_2 h_1 \mu_1^T, h_2 \mu_2^T\right],$$

it is a left eigenvector corresponding to eigenvalue 0 of matrix Φ. Choose ω_1 as a column vector of *P*, and then γ_1^T is a row vector of P^{-1}. In addition, matrix $-\Phi$ has only one eigenvalue 0 and the real part $Re(\lambda_i) < 0\left(i = 1, 2, \quad, n\right)$ of non-zero eigenvalues, from [26], we can have

$$e^{-\Phi t} = Pe^{Jt}P^{-1} = P\begin{pmatrix} 1 & 0 \\ 0 & e^{J't} \end{pmatrix}P^{-1}$$

where J' is the Jordan block of matrix $-\Phi$ corresponding to non-zero eigenvalue. From [27], we know

$$\lim_{t\to\infty} e^{J't} = \begin{pmatrix} 1 & 0 \\ & \\ 0 & 0 \end{pmatrix}$$

It is easy to know the solution of system (2-4) is $\xi(t) = e^{-\Phi t}\xi(0)$, then

$$\lim_{t\to\infty} e^{-\Phi t} = [\omega^T, \gamma^T]^T = \begin{pmatrix} k_1k_2\mathbf{1}_m h_1h_2\mu_1^T\hat{L} & k_2\mathbf{1}_m h_1\mu_1^T & \mathbf{1}_{n-m}h_2\mu_2^T \\ \mathbf{0}_{m\times m} & \mathbf{0}_{m\times m} & \mathbf{0}_{m\times(n-m)} \\ k_1k_2\mathbf{1}_m h_1h_2\mu_1^T\hat{L} & k_2\mathbf{1}_m h_1\mu_1^T & \mathbf{1}_{n-m}h_2\mu_2^T \end{pmatrix},$$

and then

$$\lim_{t\to\infty}\xi(t) = \begin{pmatrix} k_1k_2\mathbf{1}_m h_1h_2\mu_1^T\hat{L} & k_2\mathbf{1}_m h_1\mu_1^T & \mathbf{1}_{n-m}h_2\mu_2^T \\ \mathbf{0}_{m\times m} & \mathbf{0}_{m\times m} & \mathbf{0}_{m\times(n-m)} \\ k_1k_2\mathbf{1}_m h_1h_2\mu_1^T\hat{L} & k_2\mathbf{1}_m h_1\mu_1^T & \mathbf{1}_{n-m}h_2\mu_2^T \end{pmatrix}\begin{pmatrix} x_s(0) \\ v_s(0) \\ x_f(0) \end{pmatrix}$$

that is,

$$\lim_{t\to\infty}\begin{pmatrix} p_s \\ p_s \\ p_f \end{pmatrix} = \begin{pmatrix} k_1k_2\mathbf{1}_m h_1h_2\mu_1^T\hat{L} & k_2\mathbf{1}_m h_1\mu_1^T & \mathbf{1}_{n-m}h_2\mu_2^T \\ \mathbf{0}_{m\times m} & \mathbf{0}_{m\times m} & \mathbf{0}_{m\times(n-m)} \\ k_1k_2\mathbf{1}_m h_1h_2\mu_1^T\hat{L} & k_2\mathbf{1}_m h_1\mu_1^T & \mathbf{1}_{n-m}h_2\mu_2^T \end{pmatrix}\begin{pmatrix} x_s(0) \\ v_s(0) \\ x_f(0) \end{pmatrix} + \begin{pmatrix} h_2 \\ 0 \\ h_1 \end{pmatrix}$$

Based on $p_i = h_2x_i, q_i = h_2v_i, i \in \mathrm{I}_m$, and $p_i = h_1x_i, i \in \mathrm{I}_n/\mathrm{I}_m$, the convergence points of system(1) with protocol (2) can be obtained as

$$\lim_{t\to\infty} x_s = \frac{1}{h_2}(k_1k_2\mathbf{1}_m h_1h_2\mu_1^T\hat{L}x_s(0) + k_2\mathbf{1}_m h_1\mu_1^T v_s(0) + \mathbf{1}_m h_2\mu_2^T x_f(0)) + 1,$$

$$\lim_{t\to\infty} v_s = \mathbf{1}_m 0,$$

$$\lim_{t\to\infty} x_f = \frac{1}{h_1}(k_1k_2\mathbf{1}_{n-m} h_1h_2\mu_1^T\hat{L}x_s(0) + k_2\mathbf{1}_{n-m} h_1\mu_1^T v_s(0) + \mathbf{1}_{n-m} h_2\mu_2^T x_f(0)) + 1\cdot$$

Corollary 1 When $h_1=h_2$, if the feedback gains satisfy $k_1 > (1+\frac{1}{h_2})\bar{d}_s, k_2 > 0$, then system (1) with protocol (2) can reach consensus if and only if G contains a directed spanning tree, moreover, the convergence points of system (1) with protocol (2) are

$$\lim_{t\to\infty} x = k_1k_2h_1h_2\mu_1^T\hat{L}x_s(0) + k_2h_1\mu_1^T v_s(0) + h_2\mu_2^T x_f(0),$$

$$\lim_{t\to\infty} v_s = \mathbf{1}_m 0\cdot$$

where $\bar{d}_s = \max_{i=1,\ ,m}\sum_{j=1}^{n} a_{ij}$ is maximum value of the diagonal element of corresponding Laplician matrix $\overline{L}_s$.

Remark 3 Notice that $h_1=h_2$ means that there is only one group in the heterogeneous multi-agent system.

3. Simulation

In this section, some simulation examples are presented to demonstrate the effectiveness of the theoretic results for continuous-time heterogeneous multi-agent systems.

Example 1 Consider a ten-agent undirected network with four first-order integrator agents (agents 7-10) and six second-order integrator agents (agents 1-6) in Fig. 1 and the adjacency matrix A and the Laplacian matrix L are defined as

$$A=\begin{pmatrix} 0&1&1&0&0&0&0&0&0&0\\ 1&0&0&0&1&1&0&0&0&0\\ 1&0&0&1&0&0&0&0&1&0\\ 0&0&1&0&1&0&0&0&0&0\\ 0&1&0&1&0&1&0&0&0&0\\ 0&1&0&0&1&0&0&1&0&1\\ 0&0&0&0&0&0&0&1&1&0\\ 0&0&0&0&0&1&1&0&0&1\\ 0&0&1&0&0&0&1&0&0&1\\ 0&0&0&0&0&1&0&1&1&0 \end{pmatrix}, L=\begin{pmatrix} 2&-1&-1&0&0&0&0&0&0&0\\ -1&3&0&0&-1&-1&0&0&0&0\\ -1&0&3&-1&0&0&0&0&-1&0\\ 0&0&-1&2&-1&0&0&0&0&0\\ 0&-1&0&-1&3&-1&0&0&0&0\\ 0&-1&0&0&-1&4&0&-1&0&-1\\ 0&0&0&0&0&0&2&-1&-1&0\\ 0&0&0&0&0&-1&-1&3&0&-1\\ 0&0&-1&0&0&0&-1&0&3&-1\\ 0&0&0&0&0&-1&0&-1&-1&3 \end{pmatrix}.$$

For the continuous-time heterogeneous multi-agent system, we choose h_1=1, h_2=6, k_1=0.1, k_2=0.1, x_0=［2,5,3,-1,1.5,-3,10,-2.8,2,4］, v_0=［2.5,8,3,-9,-4,1］. By calculating, the conditions of Theorem 1 are satisfied, then such system can achieve consensus from Figs. 2-3, where Fig. 2 is position trajectories and Fig. 3 is velocity trajectories, respectively. Specially, Fig. 4 show the position states of the agents reach consensus asymptotically when h_1=h_2＞0.

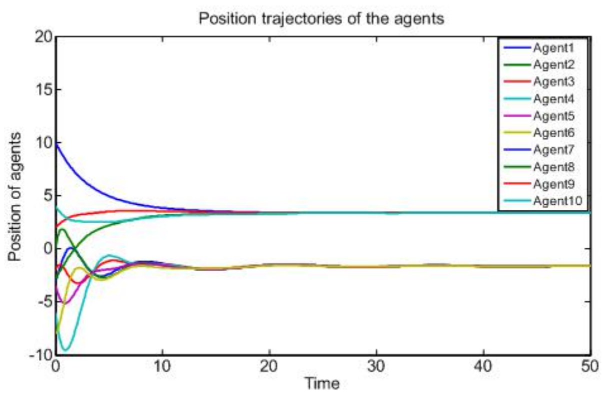

Fig. 2: Position trajectories of heterogeneous multi-agent system described as Fig. 1.

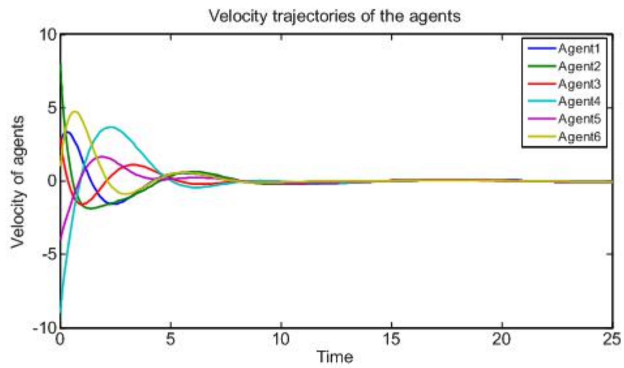

Fig. 3: Velocity trajectories of heterogeneous multi-agent system described as Fig. 1.

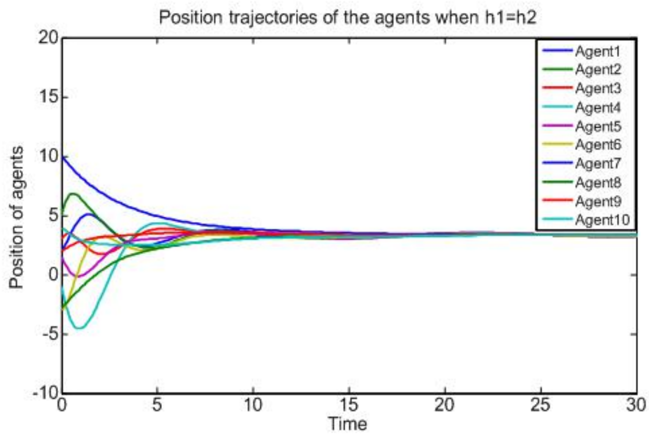

Fig. 4: Position trajectories of agents when $h_1=h_2$ in Fig. 1.

Example 2 Consider a ten-agent directed network with four first order integrator agents (agents7-10) and six second-order integrator agents (agents 1-6) in Fig. 5.

Fig. 5: A directed graph.

The adjacency matrix A and the Laplacian matrix L are defined as

$$A=\begin{pmatrix} 0&0&1&0&0&0&0&0&0&0\\ 1&0&0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&1&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&1&0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0&1&0&0 \end{pmatrix},\quad L=\begin{pmatrix} 1&0&-1&0&0&0&0&0&0&0\\ -1&2&0&0&-1&0&0&0&0&0\\ 0&0&1&-1&0&0&0&0&0&0\\ 0&0&0&1&-1&0&0&0&0&0\\ 0&0&0&0&1&-1&0&0&0&0\\ 0&-1&0&0&0&2&0&-1&0&0\\ 0&0&0&0&0&0&1&0&-1&0\\ 0&0&0&0&0&0&-1&1&0&0\\ 0&0&-1&0&0&0&0&0&2&-1\\ 0&0&0&0&0&-1&0&-1&0&2 \end{pmatrix}.$$

For the continuous-time heterogeneous multi-agent system, we choose h_1=1, h_2=6, k_1=3, k_2=1, x_0=［2, 5, 3, −1, 1. 5, −3, 10, −2. 8, 2, 4］, v_0=［2. 5, 8, 3, −9, −4, 1］. By calculating, the conditions of Theorem 3 are satisfied, then such system can achieve consensus from Figs. 6-7, where Fig. 6 is position trajectories and Fig. 7 is velocity trajectories, respectively. Specially, Fig. 8 show the position states of the agents reach consensus asymptotically when h_1=h_2＞0.

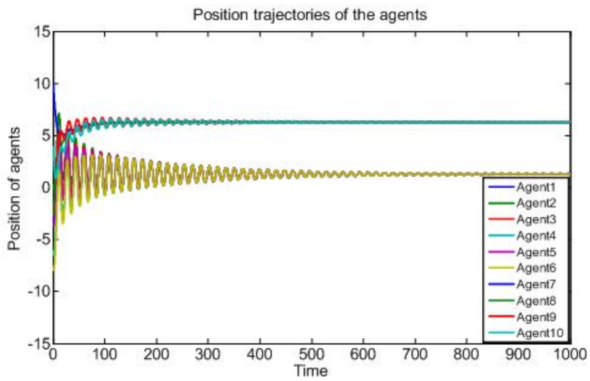

Fig.6: Position trajectories of heterogeneous multi-agent system described as Fig. 5.

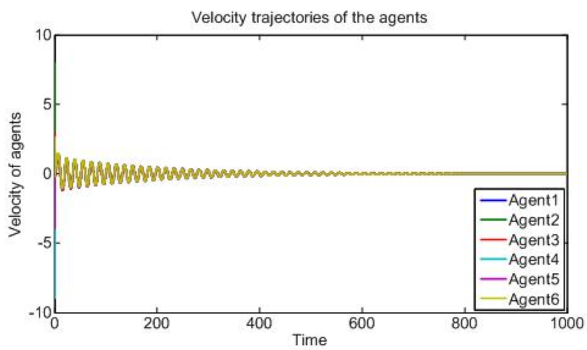

Fig. 7: Velocity trajectories of heterogeneous multi-agent system described as Fig. 5.

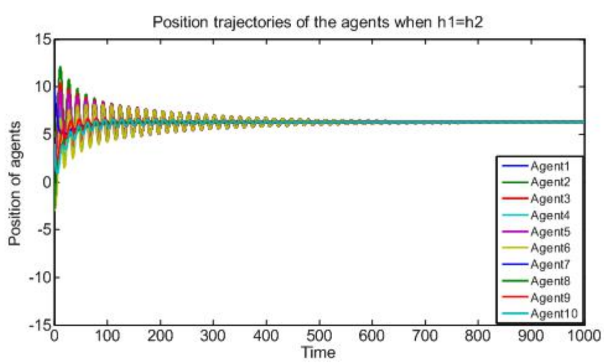

Fig. 8: Position trajectories of agents when $h_1 = h_2$ in Fig. 5.

Conclusion

In this paper, we have investigated the group consensus of heterogeneous multi-agent systems under undirected and directed fixed topology in continuous-time. we get the heterogeneous multi-agent systems can reach group consensus if and only if the topology contains a spanning tree, and we also find the convergence points of the two groups . Simulation results are also provided to illustrate the effectiveness of the theoretical results..

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