Lumped Mass Finite Element Method of BBM Equation on Rectangular Mesh

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Abstract: In this paper, the full-discrete approximation scheme of the lumped mass nonconforming finite element method for the nonlinear BBM equation is discussed on rectangular meshes. Firstly, we study the Crank-Nicolson full-discrete approximation scheme of the lumped mass finite element method for the discussed problem. Secondly, error analysis between the solution of the BBM equation and the solution of the approximated scheme are discussed. Without using traditional elliptic projection operator, the optimal error estimations are obtained on anisotropic meshes.

Key Words: lumped mass; BBM equation; anisotropic; rectangular meshes

Introduction

The lumped mass finite element method is a kind of modified finite element method, It has the same convergence and error estimation than the traditional finite element method, but it has a smaller amount of calculation. Therefore, the lumped mass finite element methods favored by scholars at home and abroad. It is also one of the hot topics being studied. BBM equation is an important equation in mathematical physics system. It has been studied extensively by Ben-jamin and others, as a model for unidirectional, long, dispersive waves. It has been widely used in linear optics, iso-particle physics, etc. The numerical solution of problem has been studied in, among them, the standard Galerkin method, the finite difference method and the general method are applied to this equation. Feng Minfu et al. proposed a Crank-Nicolson difference method to discretize the equation in . Khaled Omrani made a detailed analysis of the standard Galerkin method of this equation in : the space is discretized by the standard Galerkin, and the time discretization is in the Crank-Nicolson format, the convergence of the method is proved. Tan Yanmei et al. applied the mixed finite element method to this equation in , established semi-discretization and full discretization finite element format, the existence and uniqueness of the finite element solution is proved, and an error analysis is given. However, the research on the lumped mass finite element method of nonlinear BBM equations on rectangular meshes as not been reported.

The main purpose of this article is to study the lumped mass finite element method for the BBM equation on rectangular meshes, the optimal error estimate is obtained without using tra-ditional elliptic projection operator, time discretization by using Crank-Nicolson scheme. In this paper, C denotes generic positive constant independent of step sizes and not necessarily the same at each occurrence.

1 Lumped mass finite element method of BBM equation on rectangular meshes

We will consider the following nonlinear BBM equation:

$$\begin{cases} u_t - \Delta u_t = \nabla f(u), & \forall (X,t) \in \Omega \times (0,\ T], \\ u(X,\ t) = 0, & \forall (X,t) \in \partial\Omega \times (0,\ T], \\ u(X,\ 0) = u_0(X), & \forall X \in \bar{\Omega}. \end{cases} \tag{1}$$

Where $\Omega \subset R^2$ is a bounded domain with smooth boundary $\partial\Omega$ $0 < T < \infty$, and $f(u) = -((1/2)u^2 + u), X = (x, y).$ For a nonnegative integer m , let $H^m(\Omega)$ denote the usual Sobolev space of real-valued functions defined on Ω .We introduce the weak formulation of (1). Letting $u_t = Q$ by $v \in H_0^1(\Omega)$ and using the Green formula.

$$\begin{cases} (u_t, v) + a(Q, v) = (\nabla f(u), v), \\ (u_t, v) = (Q, v), \\ u(X, 0) = u_0(X). \end{cases} \tag{2}$$

where $a(Q,v) = (\nabla Q, \nabla v) = \int_\Omega \nabla Q \nabla v dX.$ Let $\Omega \subset R^2$ is a bounded rectangular region with boundaries parallel to the axes. T_h be an rectangular subdivision of Ω . $\bar{\Omega} = \bigcup_{K \in T_h} K,$

which does not need to satisfy the regularity assumption or quasi-uniform assumption. For a rectangular element K, its boundaries are parallel to the x-axes and the y-axis. The corresponding side lengths are $2h_x$ and $2h_y$,let the center of K is (x_k, y_k) ,the four vertices are

$a_1=(x_k-h_x,y_k-h_y),\ a_2=(x_k+h_x,y_k-h_y),\ a_3=(x_k+h_x,y_k+h_y),\ a_4=(x_k-h_x,y_k+h_y).$ the four sides are $l_i=\overline{a_ia_{i+1}}, i=1,2,3,4\ \ a_5=a_1$. Hypothesis $h_x\gg h_y$.

The linear function determined by the values of two nodes $u^h(X,t_n)$ and $u^h(X,t_{n+1})$ is an approximate solution of $u(X,t)$, then lumped mass nonconforming fully discrete Crank-Nicolson scheme of (2) is let $v\in V_h$, to find (u_{n+1}^h,Q_{n+1}^h), such that

$$\begin{cases}(u_{n+1}^h-u_n^h,v)_h+a_{1h}(Q_{n+1/2}^h,v)\Delta t=(\nabla f_{n+1/2},v)\Delta t,\\ ((u_{n+1}^h-u_n^h)/\Delta t,v)=((Q_{n+1}^h+Q_n^h)/2,v),\\ (u_0^h-\prod_h u_0,v)=0,\\ (Q_0^h-\prod_h\varphi,v)=0,\end{cases}\tag{3}$$

where u_0^h is an appropriate approximation to $u_0(X)$, $u_n=u(X,t_n),\ u_n^h=u^h(X,t_n),\ Q_n^h=Q^h(X,t_n)$, $u_{n+1/2}^h=(1/2)(u_n^h+u_{n+1}^h),\nabla f_{n+1/2}=(1/2)(\nabla f(u_n)+\nabla f(u_{n+1}))$.

According to the theoretical knowledge of the numerical solution of partial differential equations that problem (3) has a unique solution (u_{n+1}^h,Q_{n+1}^h).

2 Error estimates

For simplicity, let $u_n=u(X,t_n),Q_n=Q(X,t_n),\varepsilon_h(u,v)=(u,v)_h-(u,v),r_n=Q_n^h-\prod_h Q_n$,

$\rho_n=Q_n-\prod_h Q_n,\ \varepsilon_n=u_n^h-\prod_h u_n,\eta_n=u_n-\prod_h u_n, r|_{t=0}=\rho|_{t=0}=\varepsilon|_{t=0}=\eta|_{t=0}=0.$

Lemma 2.1. For all $v\in V_h$, the solution $u(X,t),Q(X,t)$ of (2), then

$$(u_{n+1}-u_n,v)_h+a_{1h}(Q_{n+1/2},v)\Delta t=(f_{n+1/2},\nabla v)\Delta t+E_n(v),\tag{4}$$

$$|E_n(v)|\le C[(\int_{t_n}^{t_{n+1}}\left\|\frac{\partial^2 f}{\partial t^2}\right\|_0^2dt)^{1/2}+(\int_{t_n}^{t_{n+1}}\left\|\partial^2Q/\partial t^2\right\|_1^2dt)^{1/2}](\Delta t)^{5/2}\|v\|_{1h}+Ch\int_{t_n}^{t_{n+1}}\|Q\|_2^2dt)^{1/2}](\Delta t)^{1/2}\|v\|_{1h}+Ch^2\int_{t_n}^{t_{n+1}}\|\partial u/\partial t\|_1^2dt)^{1/2}](\Delta t)^{1/2}\|v\|_{1h}.$$

Proof. From (2), for all $v\in V_h$, $(u_t,v)-(\Delta u_t,v)=(\nabla f(u),v)$, then using Green's formula we have

$$(u_t,v)+(\nabla Q,\nabla v)=(\nabla f(u),v)+\Gamma_h(Q,v),\tag{5}$$

for all $v\in V_h$, then $\Gamma_h(Q,v)\le Ch\|Q\|_2\|v\|_{1h}$. from (5), Integral on both sides for $t_n\le t\le t_{n+1}$,such that<<Eqn00056.wmf>> then

$$(u_{n+1}-u_n,v)+a_{1,h}(Q_{n+1/2},v)\Delta t=(\nabla f_{n+1/2},v)\Delta t+(\int_{t_n}^{t_{n+1}}(f-f_{n+1/2})dt,\nabla v)-a_{1h}(\int_{t_n}^{t_{n+1}}(Q-Q_{n+1/2})dt,v)+\int_{t_n}^{t_{n+1}}\Gamma_h(Q_{n+1/2},v)dt+\varepsilon_h(u_{n+1}-u_n,v).$$

and then $(u_{n+1}-u_n,v)_h+a_{1h}(Q_{n+1/2},v)\Delta t=(f_{n+1/2},\nabla v)\Delta t+E_n(v)$,

let $P_1(t)=[(t-t_n)/(t_{n+1}-t_n)]f(X,t_{n+1})+[(t-t_{n+1})/(t_n-t_{n+1})]f(X,t_n)$, we have $\int_{t_n}^{t_{n+1}}P_1(t)dt=\int_{t_n}^{t_{n+1}}f_{n+1/2}dt$.

Under the anisotropic meshes, for all $v\in V_h$, then $\|v\|_0\le C\|v\|_{1h}$, according to the one-dimensional linear interpolation theory and the Cauchy-Schwarz inequality, we get

$$\left|(\int_{t_n}^{t_{n+1}}(f-f_{n+1/2})dt,\nabla v)\right|\le C\left|(\int_{t_n}^{t_{n+1}}(\partial^2f/\partial t^2)(\Delta t)^2dt,\nabla v)\right|\le C(\int_{t_n}^{t_{n+1}}\left\|\partial^2f/\partial t^2\right\|_0^2dt)^{1/2}(\Delta t)^{5/2}\|v\|_{1h}.$$

there holds $\left|a_{1h}(\int_{t_n}^{t_{n+1}}(Q-Q_{n+1/2})dt,v)\right|\le C(\int_{t_n}^{t_{n+1}}\left|\partial^2Q/\partial t^2\right|_1^2dt)^{1/2}(\Delta t)^{5/2}\|v\|_{1h}$, $\left|\int_{t_n}^{t_{n+1}}\Gamma(Q,v)dt\right|\le Ch(\int_{t_n}^{t_{n+1}}\|Q\|_2^2dt)^{1/2}(\Delta t)^{1/2}\|v\|_{1h}$,

$|\varepsilon_h(u_{n+1}-u_n,v)|\le Ch^2\|u_{n+1}-u_n\|_1\|v\|_{1h}\le Ch^2(\int_{t_n}^{t_{n+1}}\|\partial u/\partial t\|_1^2dt)^{1/2}(\Delta t)^{1/2}\|v\|_{1h}$,

which completes the proof.

Lemma 2.2. There exists a constant r, $\Delta t<r<1$, for all L, and $1\le L\le N$ (L is a positive integer), then there holds

$$\|\varepsilon_L\|_h^2+\|\nabla r_L\|_0^2\le C[\int_0^T(\left\|\partial^2f/\partial t^2\right\|_0^2+\left\|\partial^2u/\partial t^2\right\|_1^2+\left|\partial^2Q/\partial t^2\right|_1^2+\left\|\partial^2Q/\partial t^2\right\|_1^2)dt](\Delta t)^4+Ch^2\int_0^T\|Q\|_2^2dt+Ch^4\int_0^T\|\partial u/\partial t\|_1^2dt.\tag{6}$$

Proof. Subtracting (5) from the first formula of (4), for all $v\in V_h$,we have

$$(u_{n+1}^h-u_{n+1}-(u_n^h-u_n),v)+a_{1h}(Q_{n+1/2}^h-Q_{n+1/2},v)\Delta t=-E_n(v),\tag{7}$$

according to the definition of $\varepsilon_n,r_{n+1/2}$ and η_n, from (7), we obtain

$$(\varepsilon_{n+1}-\varepsilon_n,v)_h+(\eta_{n+1}-\eta_n,v)_h+a_{1h}(r_{n+1/2},v)\Delta t-a_{1h}(Q_{n+1/2}-\prod_h Q_{n+1/2},v)\Delta t=-E_n(v),$$

by the characteristics of the unit of C-R,then $a_{1h}(Q_{n+1/2}-\prod_h Q_{n+1/2},v)=0$, further, using the second formula of (3), we get

$(\varepsilon_{n+1}-\varepsilon_n)/2=(r_{n+1}-r_n)/\Delta t+\prod_h(u_{n+1}-u_n)/\Delta t-\prod_h(Q_{n+1}-Q_n)/2,$

let $v=\varepsilon_{n+1}+\varepsilon_n$, substitute the above formula in (7), then

$$\|\varepsilon_{n+1}\|_h^2-\|\varepsilon_n\|_h^2+a_{1h}(r_{n+1/2},2(r_{n+1}-r_n))=-E_n(\varepsilon_{n+1}+\varepsilon_n)+(\eta_{n+1}-\eta_n,\varepsilon_{n+1}+\varepsilon_n)-a_{1h}(r_{n+1/2},2\prod_h((Q_{n+1}-Q_n)/\Delta t-(u_{n+1}+u_n)/2))\Delta t, \quad (8)$$

Now we shall respectively estimate the terms at the right end of the equation (8), from lemma 2.2 and Young inequality, we get

$$|E_n(\varepsilon_{n+1}+\varepsilon_n)|\le C[\int_{t_n}^{t_{n+1}}(\|\partial^2 f/\partial t^2\|_0^2+\|\partial^2 u/\partial t^2\|_1^2+|\partial^2 Q/\partial t^2|_1^2+\|\partial^2 u/\partial t^2\|_0^2)dt](\Delta t)^4+Ch^2\int_{t_n}^{t_{n+1}}\|Q\|_2^2 dt+Ch^4\int_{t_n}^{t_{n+1}}\|\partial u/\partial t\|_1^2 dt+(1/4)\|\varepsilon_{n+1}+\varepsilon_n\|_h^2\Delta t.$$

Above the second item, we obtain $|(\eta_{n+1}-\eta_n,\varepsilon_{n+1}+\varepsilon_n)_h|\le Ch^2\int_{t_n}^{t_{n+1}}|\partial u/\partial t|_1^2 dt+(1/4)\|\varepsilon_{n+1}+\varepsilon_n\|_h^2\Delta t.$

From $\|r_{n+1/2}\|_0\le C\|\nabla r_{n+1/2}\|_0$, the third item is estimated as follows

$$|a_{1h}(r_{n+1/2},2\prod_h((Q_{n+1}-Q_n)/\Delta t-(u_{n+1}+u_n)/2))\Delta t|\le 2\|\nabla r_{n+1/2}\|_0\|\nabla\prod((Q_{n+1}-Q_n)/\Delta t-(u_{n+1}+u_n)/2)\|_0\Delta t$$

$$\le C\|\nabla r_{n+1/2}\|_0\left\|\nabla\int_{t_n}^{t_{n+1}}(\partial Q/\partial t-u_{n+1/2})dt\right\|_0\le(1/2)(\|\nabla r_{n+1}\|_0^2+\|\nabla r_n\|_0^2)\Delta t+C(\int_{t_n}^{t_{n+1}}|\partial^2 u/\partial t^2|_1^2 dt)(\Delta t)^4,$$

Substitute the above estimation results in (8), furthermore, from $(1/2)\|\varepsilon_{n+1}+\varepsilon_n\|_h^2\le\|\varepsilon_{n+1}\|_h^2+\|\varepsilon_n\|_h^2$,

We have $(1-\Delta t)(\|\varepsilon_{n+1}\|_h^2+\|r_{n+1}\|_h^2)-(1+\Delta t)(\|\varepsilon_n\|_h^2+\|r_n\|_h^2)\le\theta_n$, Where

$$\theta_n=C[\int_{t_n}^{t_{n+1}}(\|\partial^2 f/\partial t^2\|_0^2+\|\partial^2 u/\partial t^2\|_1^2+|\partial^2 Q/\partial t^2|_1^2+\|\partial^2 u/\partial t^2\|_0^2)dt](\Delta t)^4+Ch^2\int_{t_n}^{t_{n+1}}\|Q\|_2^2 dt+Ch^4\int_{t_n}^{t_{n+1}}\|\partial u/\partial t\|_1^2 dt+Ch^4\Delta t.$$

For $0<1/(1+\Delta t)<1$, so that $(1-\Delta t)/(1+\Delta t)(\|\varepsilon_{n+1}\|_h^2+\|r_{n+1}\|_h^2)-(\|\varepsilon_n\|_h^2+\|r_n\|_h^2)\le\theta_n$,

$(1-\Delta t)^n/(1+\Delta t)^n$ multiplied with both sides, then summing up from $n=0$ to $L-1$, we get

$\|\varepsilon_L\|_h^2+\|\nabla r_L\|_0^2\le((1+\Delta t)/(1-\Delta t))^L\sum_{n=0}^{N-1}\theta_n$, for $((1+\Delta t)/(1-\Delta t))^L\le((1+\Delta t)/(1-\Delta t))^N=(1+(2\Delta t)/(1-\Delta t))^N<(1+(2\Delta t)/(1-r))^N\le e^{(2)/(1-r)}$, then

$$\sum_{n=0}^{N-1}\theta_n\le C[\int_0^T(\|\partial^2 f/\partial t^2\|_0^2+\|\partial^2 u/\partial t^2\|_1^2+|\partial^2 Q/\partial t^2|_1^2+\|\partial^2 Q/\partial t^2\|_1^2)dt](\Delta t)^4+Ch^2\int_0^T\|Q\|_2^2 dt+Ch^4\int_0^T\|\partial u/\partial t\|_1^2 dt, \quad (9)$$

From (9) to complete the rest of the proof.

Theorem 2.1. let $u(X,t)$ and $Q(X,t)$ be the solutions of (2), suppose $f(u)$ is sufficiently smooth, then there holds

$$\max_{1\le n\le N}\{\|u_n^h-u_n\|_0^2+\|\nabla(Q_n^h-Q_n)\|_0^2\}\le C(h^2+(\Delta t)^4).$$

Proof. Using the definition of $\varepsilon_n,\eta_n,r_n,\rho_n$ and the triangle inequality, we get $\|u_n^h-u_n\|_0^2=\|\varepsilon_n-\eta_n\|_0^2\le C(\|\varepsilon_n\|_0^2+\|\eta_n\|_0^2)\le C(\|\varepsilon_n\|_h^2+\|\eta_n\|_h^2)$, and $\|\nabla(Q_n^h-Q_n)\|_0^2\le\|\nabla r_n\|_0^2+\|\nabla\rho_n\|_0^2$, then using lemma 2.2 and the interpolation theorem, the proof is completed.

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