

The Continuity of Fuzzy Metric Spaces

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Abstract: In the present paper, the definitions of a fuzzy continuous function and uniformly fuzzy continuous are introduced. We prove that a function f from a fuzzy metric space $(\tilde{F}, \tilde{\mathcal{M}}_{\tilde{F}})$ into a fuzzy metric space $(\tilde{G}, \tilde{\mathcal{M}}_{\tilde{G}})$ is fuzzy continuous if and only if for every fuzzy open subset \tilde{A} of \tilde{G} , $f^{-1}(\tilde{A})$ is fuzzy open in \tilde{F} . Also the composition function of two uniformly fuzzy continuous functions is proved to be a uniformly fuzzy continuous function.

Keywords: Fuzzy Metric Space; Fuzzy Continuous Function; Uniformly Fuzzy Continuous Function

1. Introduction

The foundation of the concept of the fuzzy metrics is given for the first time by Kramosil and Michalek in 1975^[1] to enforce the fuzziness principle to the traditional concepts of metric and metric spaces to introduce the definition of the fuzzy metrics space by generalizing the notion of the probabilistic metric space to the fuzzy case. Kaleva and Seikkala in 1984^[2] generalized the idea of a metric space to introduce the definition of the fuzzy metrics space by specifying the distance between two points (rather than the probabilistic metric space) to be a non-negative fuzzy number. Sostak in^[3] presented an alternative method to introduce a new version of the concept of a fuzzy metric called revised fuzzy metric. A t-conorm binary operation is used in the definition of the revised fuzzy metric to assess the degree of proximity of two points. Moreover, many researchers had different views on the problem of constructing a fuzzy metric space. In particular, the concept of fuzzy metric space given in^[1] is modified by George and Veeramani^[4] in terms of Hausdorff topology. In addition, Xie *et al.*^[5] studied the relation between the fuzzy measure and the fuzzy metric space. Authors in^[6] gave a new definition of fuzzy metric space by using fuzzy scalars instead of

fuzzy numbers or real numbers and proved basic theories about this space. In^[7] Gupta and Kanwar have made efforts to introduce V-fuzzy metric spaces and to study their main properties. Other notations and approaches for fuzzy metric spaces are considered in^[8-11].

The aim of this paper is to introduce the definition of a fuzzy continuous function and uniformly fuzzy continuous function in a fuzzy metric space (FM-space) $(\tilde{F}, \tilde{\mathcal{M}})$ given in^[12] and proved essentially theorems.

Several researcher can calculate the properties of the materials in applied physics using mathematical models by means of these theorems^[13-91].

The structure of this paper is as follows. In section 2 some properties and basic notions of the fuzzy metric space (FM-space) are given. The concepts of a fuzzy continuous and uniformly fuzzy continuous function are introduced in section 3, moreover, some important properties of the given definitions are investigated.

2. Preliminaries

In this section, we restate basic results and some definitions.

Definition 2.1^[92]

A fuzzy metric space (briefly, FM-space) is an ordered pair $(\tilde{F}, \tilde{\mathcal{M}})$ where \tilde{F} is a fuzzy set and $\tilde{\mathcal{M}}$ is a mapping from $\tilde{F} \times \tilde{F} \times (0,1]$ into $I = [0,1]$ such that the following five properties hold, for each $(a, \alpha), (b, \beta), (c, \delta) \in \tilde{F}$:

- ($\tilde{\mathcal{M}}1$) $\tilde{\mathcal{M}}(a, b, \gamma) > 0$ if $a \neq b$ where $\gamma = \max\{\alpha, \beta\}$
- ($\tilde{\mathcal{M}}2$) $\tilde{\mathcal{M}}(a, b, \gamma) = 0$ if and only if $a = b$.
- ($\tilde{\mathcal{M}}3$) $\tilde{\mathcal{M}}(a, b, \gamma) = \tilde{\mathcal{M}}(b, a, \gamma)$.
- ($\tilde{\mathcal{M}}4$) $\tilde{\mathcal{M}}(a, b, \gamma) \leq \tilde{\mathcal{M}}(a, c, \gamma) + \tilde{\mathcal{M}}(c, b, \gamma)$, where $\gamma = \max\{\alpha, \beta, \delta\}$
- ($\tilde{\mathcal{M}}5$) If $0 < \lambda \leq \gamma < 1$ then $\tilde{\mathcal{M}}(a, 0, \gamma) \leq \tilde{\mathcal{M}}(a, 0, \lambda)$ and there exists $0 < \gamma_n < \gamma$ such that $\lim_{n \rightarrow \infty} \tilde{\mathcal{M}}(a_n, 0, \gamma_n) = \tilde{\mathcal{M}}(a, 0, \gamma)$.

Definition 2.2^[92]

Let $(\tilde{F}, \tilde{\mathcal{M}})$ be an FM-space, and let $(a, \alpha) \in \tilde{F}$, where $\alpha \in (0,1]$. Given real number $\varepsilon > 0$, then:

- (1) $\tilde{O}_\varepsilon(a_1, \alpha_1) = \{(a, \alpha) \in \tilde{F} : \tilde{\mathcal{M}}(a, a_1, \gamma) < \varepsilon\}$ is called the fuzzy open ball of radius ε where $\gamma = \max\{\alpha, \alpha_1 \in (0,1]\}$.
- (2) $\tilde{B}_\varepsilon[a_1, \alpha_1] = \{(a, \alpha) \in \tilde{F} : \tilde{\mathcal{M}}(a, a_1, \gamma) \leq \varepsilon\}$ is called the fuzzy closed ball of radius ε .

Definition 2.3^[92]

Let $(\tilde{F}, \tilde{\mathcal{M}})$ be an FM-space. A fuzzy subset $\tilde{A} \subseteq \tilde{F}$ is fuzzy open if and only if there exists a fuzzy open ball $\tilde{O}_\varepsilon(a, \alpha)$ centered at every fuzzy point (a, α) in \tilde{A} that are contained in \tilde{A} . A fuzzy subset $\tilde{B} \subseteq \tilde{F}$ is called fuzzy closed if $\tilde{B}^c = \tilde{F} - \tilde{B}$ is fuzzy open.

Definition 2.4^[92]

In an FM-space $(\tilde{F}, \tilde{\mathcal{M}})$, a fuzzy sequence $\{(a_n, \alpha_n)\}$ where $\alpha, \alpha_n \in (0,1]$ is said to be

- (i) Convergent if there exists $(a, \alpha) \in \tilde{F}$ such that $\lim_{n \rightarrow \infty} \tilde{\mathcal{M}}(a_n, a, \gamma) = 0$ where $\gamma = \max\{\alpha_n, \alpha\}$ or simply written $\{(a_n, \alpha_n)\} \rightarrow (a, \alpha)$.
- (ii) Cauchy if for all $\varepsilon > 0$ there is an integer number $N \in \mathbb{N}$ such that $\tilde{\mathcal{M}}(a_n, a_m, \gamma) < \varepsilon$ for every $n, m \geq N$ where $\gamma = \max\{\alpha_n, \alpha_m\}$.

Definition 2.5^[92]

A fuzzy set \tilde{A} in an FM-space $(\tilde{F}, \tilde{\mathcal{M}})$ is called fuzzy bounded if there exists $0 < r < 1$ such that $\tilde{\mathcal{M}}(a, b, \gamma) < r$. r call for each $(a, \alpha), (b, \beta) \in \tilde{F}$, $\gamma = \max\{\alpha, \beta\}$.

Definition 2.6^[92]

A fuzzy sequence $\{(a_n, \alpha_n)\}$ in an FM-space

$(\tilde{F}, \tilde{\mathcal{M}})$ is said to be fuzzy bounded if the corresponding fuzzy set is fuzzy bounded.

Definition 2.7^[92]

An FM-space $(\tilde{F}, \tilde{\mathcal{M}})$ is called complete if every Cauchy fuzzy sequence in \tilde{F} is a fuzzy convergent.

Definition 2.8^[92]

Let $(\tilde{F}, \tilde{\mathcal{M}})$ be an FM-space and let $\{(a_n, \alpha_n)\}_{n \geq 1}$ be a fuzzy sequence of real numbers. Given $r_1 < r_2 < \dots < r_n \dots$ be strictly increasing sequence of natural numbers. Then $\{(a_{nk}, \alpha_{nk})\}_{n \geq 1}$ is called a fuzzy subsequence of $\{(a_n, \alpha_n)\}_{n \geq 1}$.

Definition 2.9^[12]

Let U be a universal set, then for any $\alpha \in (0,1]$ and $u \in U$, a fuzzy subset u_α of U is called a fuzzy point in U if $u_\alpha(w) = \begin{cases} \alpha & \text{if } u = w \\ 0 & \text{otherwise} \end{cases}$ for each $w \in U$.

Now, the definition of a fuzzy metric space is given.

3. Fuzzy continuous and uniformly fuzzy continuous function on fuzzy metric space

In this section, the continuity of the fuzzy metric space is discussed. So the definition of the fuzzy continuous function at a fuzzy point in the fuzzy metric space is introduced initially.

Definition 3.1

Let $(\tilde{F}, \tilde{\mathcal{M}}_{\tilde{F}})$ and $(\tilde{G}, \tilde{\mathcal{M}}_{\tilde{G}})$ be an FM-spaces and let $\tilde{A} \subseteq \tilde{F}$. Then the function $f: \tilde{A} \rightarrow \tilde{G}$ is said to be fuzzy continuous function at a fuzzy point $(a, \alpha) \in \tilde{A}$ if for every $\varepsilon > 0$ there exists $\sigma > 0$ such that $\tilde{\mathcal{M}}_{\tilde{G}}(f(b), f(a), \gamma) < \varepsilon$ whenever $(b, \beta) \in \tilde{A}$ implies $\tilde{\mathcal{M}}_{\tilde{F}}(b, a, \gamma) < \sigma$.

The next theorem gives a characterization of the fuzzy continuous function.

Theorem 3.2

Suppose that $(\tilde{F}, \tilde{\mathcal{M}}_{\tilde{F}})$ and $(\tilde{G}, \tilde{\mathcal{M}}_{\tilde{G}})$ be two FM-spaces and let $\tilde{A} \subseteq \tilde{F}$. The function $f: \tilde{A} \rightarrow \tilde{G}$ is a fuzzy continuous function at a fuzzy point $(a, \alpha) \in \tilde{A}$ if and only if the sequence $\{(f(b_n, \alpha_n))\}$ fuzzy converges to $f(a, \alpha)$ for any fuzzy sequence of fuzzy points $\{(b_n, \alpha_n)\}$ in \tilde{A} that fuzzy converges to (a, α) .

Proof

Assume that the function $f: \tilde{A} \rightarrow \tilde{G}$ is fuzzy continuous at a fuzzy point (a, α) in \tilde{A} . Let $\{(b_n, \alpha_n)\}$ be a

fuzzy sequence in \tilde{A} that is fuzzy converges to (a, α) . Given $\varepsilon > 0$ and by the continuity of f there is $\sigma > 0$ such that $\tilde{\mathcal{M}}_{\tilde{G}}(f(b), f(a), \gamma) < \varepsilon$ whenever $(b, \beta) \in \tilde{A}$ satisfy $\tilde{\mathcal{M}}_{\tilde{F}}(b, a, \gamma) < \sigma$. Since the fuzzy sequence $\{(b_n, \alpha_n)\}$ converges to (a, α) then we can find a number $n \geq N$ with $\tilde{\mathcal{M}}_{\tilde{F}}(b_n, a, \gamma) < \sigma$. Hence for $n \geq N$ implies $\tilde{\mathcal{M}}_{\tilde{G}}(f(b_n), f(a), \gamma) < \varepsilon$, where $\gamma = \max\{\alpha_n, \alpha \in (0, 1)\}$. Thus the fuzzy sequence $\{(f(b_n, \alpha_n))\}$ fuzzy converges to $f(a, \alpha)$. For the converse, assume that every fuzzy sequence $\{(b_n, \alpha_n)\}$ in \tilde{A} converging to (a, α) has the property that the fuzzy sequence $\{(f(b_n, \alpha_n))\}$ fuzzy converges to $f(a, \alpha)$. We shall prove that f is fuzzy continuous at (a, α) . We claim f is not fuzzy continuous at (a, α) . Then there exists some $\varepsilon > 0$ and for which no $\sigma > 0$ can satisfy the requirement that $(b, \beta) \in \tilde{A}$ and $\tilde{\mathcal{M}}_{\tilde{F}}(b, a, \gamma) < \sigma$ implies $\tilde{\mathcal{M}}_{\tilde{G}}(f(b), f(a), \gamma) < \varepsilon$, that means for each $\sigma > 0$ there is a fuzzy point $(b, \beta) \in \tilde{A}$ with $\tilde{\mathcal{M}}_{\tilde{F}}(b, a, \gamma) < \sigma$ but $\tilde{\mathcal{M}}_{\tilde{G}}(f(b), f(a), \gamma) \geq \varepsilon$. Now for each $n \in N$, there is a fuzzy point $(b_n, \alpha_n) \in \tilde{A}$ with $\tilde{\mathcal{M}}_{\tilde{F}}(b_n, a, \gamma) \leq (1 - \frac{1}{n})$ but $\tilde{\mathcal{M}}_{\tilde{G}}(f(b_n), f(a), \gamma) \geq \varepsilon$. Then the fuzzy sequence $\{(b_n, \alpha_n)\}$ fuzzy converges to fuzzy point (a, α) but the fuzzy sequence $\{(f(b_n, \alpha_n))\}$ does not fuzzy converges to $f(a, \alpha)$. This contradicts the assumption that each sequence $\{(b_n, \alpha_n)\}$ in \tilde{A} fuzzy converging to (a, α) with the property $\{(f(b_n, \alpha_n))\}$ fuzzy converging to $f(a, \alpha)$. Hence our claim that f is not fuzzy continuous at (a, α) must be false.

One more characterization for the fuzzy continuous function is assigned in the following result.

Theorem 3.3

Let $(\tilde{F}, \tilde{\mathcal{M}}_{\tilde{F}})$ and $(\tilde{G}, \tilde{\mathcal{M}}_{\tilde{G}})$ be two FM-spaces. A function $f: \tilde{F} \rightarrow \tilde{G}$ is fuzzy continuous at $(a, \alpha) \in \tilde{F}$ if and only if for every $\varepsilon > 0$ there exists $\sigma > 0$ such that $\tilde{O}_\sigma(a, \alpha) \subseteq f^{-1}[\tilde{O}_\varepsilon(a, \alpha)]$.

Proof

The function $f: \tilde{F} \rightarrow \tilde{G}$ is fuzzy continuous at $(a, \alpha) \in \tilde{F}$ if and only if for every $\varepsilon > 0$ there exists $\sigma > 0$ with $\tilde{\mathcal{M}}_{\tilde{G}}(f(b), f(a), \gamma) < \varepsilon$ for each $(b, \beta) \in \tilde{F}$ implies $\tilde{\mathcal{M}}_{\tilde{F}}(b, a, \gamma) < \sigma$ and this mean $(b, \beta) \in \tilde{O}_\sigma(a, \alpha)$ satisfying $f(b, \beta) \in \tilde{O}_\varepsilon(f(a, \alpha))$ or $f[\tilde{O}_\sigma(a, \alpha)] \subseteq \tilde{O}_\varepsilon(f(a, \alpha))$. Hence $\tilde{O}_\sigma(a, \alpha) \subseteq f^{-1}[\tilde{O}_\varepsilon(a, \alpha)]$.

According to the previous theorem, the following corollary is proved.

Corollary 3.4

A function $f: \tilde{F} \rightarrow \tilde{G}$ is fuzzy continuous on \tilde{F} if and only if for every fuzzy open subset \tilde{A} of \tilde{G} , $f^{-1}(\tilde{A})$ is fuzzy open in \tilde{F} where $(\tilde{F}, \tilde{\mathcal{M}}_{\tilde{F}})$ and $(\tilde{G}, \tilde{\mathcal{M}}_{\tilde{G}})$ are FM-spaces.

Proof

Assume that f is a fuzzy continuous function and let \tilde{A} be a fuzzy open set in \tilde{G} . We prove that $f^{-1}(\tilde{A})$ is fuzzy open in \tilde{F} . Since \emptyset and \tilde{F} are fuzzy open, we may assume that $f^{-1}(\tilde{A}) \neq \emptyset$ and $f^{-1}(\tilde{A}) \neq \tilde{F}$. Consider $(b, \beta) \in f^{-1}(\tilde{A})$ then $f(b, \beta) \in \tilde{A}$. By assumption, \tilde{A} is fuzzy open so there is and may assume that n ubset t $\varepsilon > 0$ such that $\tilde{O}_\varepsilon(f(b, \beta)) \subseteq \tilde{A}$, since f is fuzzy continuous at (b, β) by Theorem (3.3) there is some $\sigma > 0$ with $\tilde{O}_\sigma(b, \beta) \subseteq f^{-1}[\tilde{O}_\varepsilon(b, \beta)] \subseteq f^{-1}(\tilde{A})$. Hence each fuzzy point (b, β) of $f^{-1}(\tilde{A})$ is an interior fuzzy point and so $f^{-1}(\tilde{A})$ is a fuzzy open in \tilde{F} .

Conversely, suppose that $f^{-1}(\tilde{A})$ is a fuzzy open in \tilde{F} for any fuzzy open set \tilde{A} of \tilde{G} . Now for each $\varepsilon > 0$, the fuzzy ball $\tilde{O}_\varepsilon(f(b, \beta))$ where $(b, \beta) \in \tilde{F}$ is fuzzy open in \tilde{F} . Since $(b, \beta) \in f^{-1}[\tilde{O}_\varepsilon(b, \beta)]$ its follows that there is some $\sigma > 0$ with $\tilde{O}_\sigma(b, \beta) \subseteq f^{-1}[\tilde{O}_\varepsilon(b, \beta)]$ and by Theorem (3.3) concludes that f is fuzzy continuous.

Corollary 3.5

Let $(\tilde{F}, \tilde{\mathcal{M}}_{\tilde{F}})$ and $(\tilde{G}, \tilde{\mathcal{M}}_{\tilde{G}})$ be an FM-spaces. A function $f: \tilde{F} \rightarrow \tilde{G}$ is fuzzy continuous on \tilde{F} if and only if $f^{-1}(\tilde{A})$ is fuzzy closed in \tilde{F} for each fuzzy closed subset \tilde{A} of \tilde{G} .

Proof

Consider \tilde{A} be a fuzzy closed subset of \tilde{G} then $\tilde{G} - \tilde{A}$ is fuzzy open in \tilde{G} therefore $f^{-1}(\tilde{G} - \tilde{A})$ is fuzzy open in \tilde{F} by Corollary (4.4). But $f^{-1}(\tilde{G} - \tilde{A}) = \tilde{F} - f^{-1}(\tilde{A})$ so $f^{-1}(\tilde{A})$ is fuzzy closed in \tilde{F} .

Conversely, assume that $f^{-1}(\tilde{A})$ is fuzzy closed in \tilde{F} for each fuzzy closed subset \tilde{A} of \tilde{G} . But the whole space \tilde{F} and \emptyset are fuzzy closed set, then $\tilde{F} - f^{-1}(\tilde{A})$ is fuzzy open in \tilde{F} and $f^{-1}(\tilde{G} - \tilde{A}) = \tilde{F} - f^{-1}(\tilde{A})$ is fuzzy open in \tilde{F} . Since each fuzzy open subset of \tilde{G} is of the type $\tilde{G} - \tilde{A}$, where \tilde{A} is a fuzzy closed set and by using Corollary (3.4) it follows that f is fuzzy continuous.

The following theorem demonstrates the composition function of two fuzzy continuous functions must be a fuzzy continuous function.

Proposition 3.6

Let $(\tilde{F}, \tilde{\mathcal{M}}_{\tilde{F}})$, $(\tilde{G}, \tilde{\mathcal{M}}_{\tilde{G}})$ and $(\tilde{P}, \tilde{\mathcal{M}}_{\tilde{P}})$ be three

FM-spaces and let $f: \tilde{F} \rightarrow \tilde{G}$, $h: \tilde{G} \rightarrow \tilde{P}$ be fuzzy continuous functions. Then the composition function $(h \circ f): \tilde{F} \rightarrow \tilde{P}$ is a fuzzy continuous function.

Proof

Let \tilde{A} be a fuzzy open subset of \tilde{P} . By Corollary (4.4) $h^{-1}(\tilde{A})$ is a fuzzy open subset of \tilde{G} . Again by Corollary (3.4), we get $f^{-1}(h^{-1}(\tilde{A}))$ is a fuzzy open subset of \tilde{F} , since $(h \circ f)^{-1}(\tilde{A}) = f^{-1}(h^{-1}(\tilde{A}))$ and from Corollary (3.4) we conclude that $(h \circ f)$ is a fuzzy continuous function.

The concept of uniformly fuzzy continuous function in an FM-space is introduced in the following definition.

Definition 3.7

Let $(\tilde{F}, \tilde{\mathcal{M}}_{\tilde{F}})$ and $(\tilde{G}, \tilde{\mathcal{M}}_{\tilde{G}})$ be two FM-spaces. A function $f: \tilde{F} \rightarrow \tilde{G}$ is said to be uniformly fuzzy continuous on \tilde{F} if for each $\varepsilon > 0$ there is some $\sigma > 0$ such that $\tilde{\mathcal{M}}_{\tilde{G}}(f(a_1), f(a_2), \gamma) < \varepsilon$ whenever $\tilde{\mathcal{M}}_{\tilde{F}}(a_1, a_2, \gamma) < \sigma$.

The Cauchy fuzzy property of the sequence in an FM-space discusses in the following theorem.

Theorem 3.8

Suppose that $(\tilde{F}, \tilde{\mathcal{M}}_{\tilde{F}})$ and $(\tilde{G}, \tilde{\mathcal{M}}_{\tilde{G}})$ are two FM-spaces and let $f: \tilde{F} \rightarrow \tilde{G}$ be a uniformly fuzzy continuous function. If $\{(a_n, \alpha_n)\}$ is a Cauchy fuzzy sequence in \tilde{F} then $\{(f(a_n, \alpha_n))\}$ is Cauchy fuzzy sequence in \tilde{G} .

Proof

Let f be uniformly fuzzy continuous. Then by definition(3.7) for each $\varepsilon > 0$ there is $\sigma > 0$ such that $\tilde{\mathcal{M}}_{\tilde{G}}(f(a_1), f(a_2), \gamma) < \varepsilon$ whenever $\tilde{\mathcal{M}}_{\tilde{F}}(a_1, a_2, \gamma) < \sigma$, for each $(a_1, \alpha_1), (a_2, \alpha_2) \in \tilde{F}$, where $\gamma = \max\{\alpha_1, \alpha_2 \in (0,1]\}$. Since $\{(a_n, \alpha_n)\}$ is a Cauchy fuzzy sequence in \tilde{F} corresponding to $\sigma > 0$ there is an integer number $N \in \mathbb{N}$ with $\tilde{\mathcal{M}}_{\tilde{F}}(a_n, a_m, \gamma) < \sigma$ for any $m, n \geq N$. Hence we conclude that $\tilde{\mathcal{M}}_{\tilde{G}}(f(a_n), f(a_m), \gamma) < \varepsilon$ for each $m, n \geq N$ and this implies that $\{(f(a_n, \alpha_n))\}$ is a Cauchy fuzzy sequence.

The following theorem demonstrates the composition function of two uniformly fuzzy continuous functions must be a uniformly fuzzy continuous function.

Proposition 3.9

Let $(\tilde{F}, \tilde{\mathcal{M}}_{\tilde{F}})$, $(\tilde{G}, \tilde{\mathcal{M}}_{\tilde{G}})$ and $(\tilde{P}, \tilde{\mathcal{M}}_{\tilde{P}})$ be an FM-spaces. Let $f: \tilde{F} \rightarrow \tilde{G}$ and $h: \tilde{G} \rightarrow \tilde{P}$. Then if f is uniformly fuzzy continuous on \tilde{F} and h is uniformly fuzzy continuous on $f(\tilde{F})$ then $(h \circ f): \tilde{F} \rightarrow \tilde{P}$ is uniformly fuzzy continuous on \tilde{F} .

Proof

Let $\varepsilon > 0$ be given. We want to find a $\sigma > 0$ such that if $(a_1, \alpha_1), (a_2, \alpha_2) \in \tilde{F}$ and $\tilde{\mathcal{M}}_{\tilde{F}}(a_1, a_2, \gamma) < \sigma$ then $\tilde{\mathcal{M}}_{\tilde{P}}(h(f(a_1)), h(f(a_2)), \gamma) < \varepsilon$. Since h is uniformly fuzzy continuous on $f(\tilde{F})$ then there exists a $\sigma_1 > 0$ such that if $f(a_1, \alpha_1), f(a_2, \alpha_2) \in \tilde{F}$ and $\tilde{\mathcal{M}}_{\tilde{G}}(f(a_1), f(a_2), \gamma) < \sigma_1$.

$$\text{Then, } \tilde{\mathcal{M}}_{\tilde{P}}(h(f(a_1)), h(f(a_2)), \gamma) < \varepsilon \tag{1}$$

Now, since f is uniformly fuzzy continuous on \tilde{F} then there exists a $\sigma_2 > 0$ such that if $(a_1, \alpha_1), (a_2, \alpha_2) \in \tilde{F}$ and $\tilde{\mathcal{M}}_{\tilde{F}}(a_1, a_2, \gamma) < \sigma_2$.

$$\text{Then, } \tilde{\mathcal{M}}_{\tilde{G}}(f(a_1), f(a_2), \gamma) < \sigma_1 \tag{2}$$

Put $\sigma = \sigma_2$, then for each $(a_1, \alpha_1), (a_2, \alpha_2) \in \tilde{F}$ we have that (1) holds. Since (1) holds we have that (2) holds. Thus $(h \circ f)$ is uniformly fuzzy continuous on \tilde{F} .

4. Conclusion

In this paper, the definitions of fuzzy continuous function from fuzzy metric space $(\tilde{F}, \tilde{\mathcal{M}}_{\tilde{F}})$ into a fuzzy metric space $(\tilde{G}, \tilde{\mathcal{M}}_{\tilde{G}})$ is introduced then several properties are discussed. Also the definition of uniformly fuzzy continuous function from a fuzzy metric space $(\tilde{F}, \tilde{\mathcal{M}}_{\tilde{F}})$ into a fuzzy metric space $(\tilde{G}, \tilde{\mathcal{M}}_{\tilde{G}})$ is given and essential theorems are proved.

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